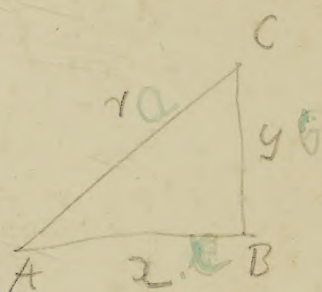


L H O O P E R

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F. Sturg



$$\text{Sine } A = \frac{y}{r}$$

$$\text{Cos } " = \frac{x}{r}$$

$$\text{Tan } A = \frac{y}{x}$$

$$\text{Sec } = \frac{r}{x}$$

$$\text{Co sec } = \frac{r}{y}$$

$$\text{Cotan } = \frac{x}{y}$$

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PREFACE

THE common examination syllabus 'Trigonometry up to and including solution of triangles' has caused most textbooks to treat the subject as if the sole use of Trigonometry were to solve triangles, and the practical examples deal almost exclusively with various forms of triangle-solving under the heading 'Heights and Distances'. Further it is customary to define the trigonometrical ratios by means of a right-angled triangle; this encourages the mistaken idea that the ratios are fundamentally attached to a triangle, and does not impress upon the pupil the fact that they are the property of an angle and of an angle only.

In this book the trigonometrical ratios are introduced as functions of the angle. The trigonometrical properties of the single angle are treated fully in the early chapters, and from the beginning the examples apply Trigonometry wherever it may be useful, to Geometry, Mensuration, Analytical Geometry, Physical formulae, &c. The right-angled triangle definitions are given in Chapter V. This chapter contains, in addition to the usual matter, a short treatment of Plane Sailing in Navigation. It is hoped that the examples in Navigation will provide practice in the use of tables, and at the same time be of interest to the pupil. Other examples

in this chapter lead up to the formulae dealing with the ordinary triangle.

The formulae for $\frac{1}{2}A$ are proved by Geometry, independently of the addition formulae; thus the triangle is treated fully without breaking the sequence with a discussion of the $A + B$ and allied formulae.

A chapter is devoted to Projection. This includes a discussion of Vector Quantities, their composition and resolution, and finishes with a geometrical treatment of Demoiivre's Theorem. In the last chapter, the addition formulae and the allied formulae are treated fully; the projection proofs are used and recommended but the old-fashioned proofs are also given.

Throughout the book the student is given every opportunity of developing the subject for himself. A large portion of the bookwork first occurs among the examples of earlier chapters. Also, when a formula has been proved, the proofs of others of the same kind are left for the student to supply. Thus, when $\sin^2 A + \cos^2 A = 1$ has been proved, the student should have no difficulty in proving the connexion between $\sec^2 A$ and $\tan^2 A$; when $\sin(A + B)$ has been found, the student should himself find the expanded form of $\cos(A + B)$, &c.

The sets of examples in the body of a chapter are numbered IV. a, IV. b, &c.; these deal only with the matter immediately preceding them. The last set of examples in a chapter has no distinguishing letter and serves for revision of the whole chapter. There are also three sets of Revision Examples. Bookwork is frequently set as an example, both in the Revision Sets and elsewhere; only by constant repetition, oral or written, can the

bookwork be learnt. There are a few sets of oral examples; these are intended to fill up spare minutes at the end of a lesson and often bring out the weak points in a pupil's knowledge. The book contains nearly 1,000 examples; it is not intended that any one should attempt all these, but it is hoped that they include a sufficient variety of types and a sufficient number of each type to meet all requirements.

Many examples are taken from Examination Papers by kind permission of the following authorities:—

The Controller of His Majesty's Stationery Office.

The University of Cambridge.

The Joint Matriculation Board of the Scottish Universities.

The Intermediate Education Board for Ireland.

The Oxford and Cambridge Schools Examination Board.

The Delegacy for Oxford Local Examinations.

The Syndicate for Cambridge Local Examinations.

The College of Preceptors.

I am indebted to Mr. Norman Chignell, B.A., of Charterhouse, for many suggestions and for assistance in correcting the proof-sheets. It is too much to hope that the answers are wholly free from mistakes, and I shall be grateful to receive early intimation of any corrections that may be found necessary.

W. E. P.

April, 1911.

CONTENTS.

CHAP.	PAGE
PRELIMINARY. PROPOSITIONS IN GEOMETRY. GRAPHS.	
LOGARITHMS. GREEK ALPHABET	9
I. ANGLES AND THEIR MEASUREMENT. THE DEGREE. POSITIVE AND NEGATIVE ANGLES. THE GRADE. CONNEXION BETWEEN ARC AND ANGLE. CIRCULAR MEASURE. THE POINTS OF THE COMPASS. GRADIENT	25
II. DEFINITIONS OF TRIGONOMETRICAL RATIOS. THEIR GRAPHS. INVERSE NOTATION. CONSTRU- TION OF ANGLE WITH GIVEN RATIO . . .	37
III. ELEMENTARY FORMULAE. IDENTITIES. RATIOS OF COMPLEMENTARY AND SUPPLEMENTARY ANGLES. RATIOS OF 0° , 90° , 30° , 60° , 45° . THE SMALL ANGLE	49
REVISION EXAMPLES A	65
IV. USE OF TABLES. HARDER GRAPHS. GENERAL SOLUTIONS OF EQUATIONS. USE OF LOGA- RITHMS	70
V. THE RIGHT-ANGLED TRIANGLE DEFINITIONS OF THE TRIGONOMETRICAL RATIOS. SOLUTION OF RIGHT-ANGLED TRIANGLES. NAVIGATION; PLANE SAILING. RATIOS OF THE DOUBLE ANGLE AND HALF ANGLE	85
VI. RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE. AREA OF TRIANGLE. THE CIRCLES OF THE TRIANGLE	107
VII. SOLUTION OF TRIANGLES	119
REVISION EXAMPLES B	134

CHAP.	PAGE
VIII. PROJECTION. VECTOR QUANTITIES. COMPOSITION AND RESOLUTION OF VECTORS. PROJECTION OF AN AREA ON A PLANE. GEOMETRICAL REPRESENTATION OF IMAGINARY QUANTITIES. DEMOIVRE'S THEOREM	139
IX. RATIO OF THE SUM OR DIFFERENCE OF TWO ANGLES. SUM OR DIFFERENCE OF TWO SINES OR COSINES. PRODUCT OF SINES AND COSINES. THE DOUBLE ANGLE AND HALF-ANGLE . . .	155
REVISION EXAMPLES C	175
MISCELLANEOUS PROBLEMS (TAKEN FROM RECENT ARMY ENTRANCE AND CIVIL SERVICE EXAMINATION PAPERS)	184
EXAMINATION PAPERS	190
INDEX	201

The following course of reading is suggested for beginners:—

- Chapter I, §§ 1-4, 9-11.
- Chapter II.
- Chapter III, §§ 21-31.
- Chapter IV, §§ 34-7, 40-2.
- Chapter V, §§ 43-5, 51-3.
- Chapter VI, §§ 54-8.
- Chapter VII.

PRELIMINARY CHAPTER

GEOMETRY

A KNOWLEDGE of the following geometrical facts is required.* In this book these propositions are referred to by the numbers given below.

Angles.

Prop. 1. If a straight line meets another straight line, the adjacent angles are together equal to two right angles.

Prop. 2. If two straight lines cut, the vertically opposite angles are equal.

Prop. 3. The angle at the centre of a circle is double an angle at the circumference standing on the same arc.

Prop. 4. Angles in the same segment of a circle are equal.

Prop. 5. Angles at the centre of a circle standing on different arcs are in the same ratio as the lengths of the arcs.

Triangles.

Prop. 6. (a) The three angles of a triangle are together equal to two right angles.

(b) If one side be produced the exterior angle equals the sum of the two interior opposite angles.

Prop. 7. Any two sides of a triangle are together greater than the third.

Prop. 8. Two triangles are congruent (i. e. are equal in every respect) if they have—

(a) two sides of the one equal to two sides of the other, each to each, and the angle contained by the two sides of the one equal to the angle contained by the two corresponding sides of the other;

or (b) three sides of the one equal to three sides of the other, each to each;

or (c) two angles of the one equal to two angles of the other,

* For proofs see Warren's *Experimental and Theoretical Geometry* (Clarendon Press), or any standard textbook.

each to each, and a side of the one equal to the corresponding side of the other.

Prop. 9. If two triangles have an angle of the one equal to an angle of the other, and the sides about another pair of angles equal, each to each, then the third angles are either equal or supplementary.

Prop. 10. (a) If two sides of a triangle are equal, the opposite angles are equal.

(b) If two sides are unequal, the greater side is opposite a greater angle.

(c) If all the sides of a triangle are equal, all the angles are equal.

Prop. 11. (a) If two angles of a triangle are equal, the opposite sides are equal.

(b) If two angles are unequal, the greater angle is opposite a greater side.

(c) If all the angles of a triangle are equal, all the sides are equal.

Prop. 12. Two triangles are similar (i. e. their angles are equal, each to each, and the ratio of pairs of sides opposite equal angles is the same for all three angles) if they have—

(a) their angles equal each to each;

(b) their sides in the same ratio;

(c) an angle of the one equal to an angle of the other, and the sides about the equal angles in the same ratio.

Prop. 13. (Pythagoras' Theorem.) In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Parallel Lines.

Prop. 14. (a) If a line is drawn to cut two parallel lines, it makes (i) the alternate angles equal, (ii) the interior angles on the same side of it together equal to two right angles, (iii) the exterior angle equal to the interior opposite angle.

(b) The opposite sides and angles of a parallelogram are equal.

Area.

The unit of area is the area of a square whose side is of unit length.

Prop. 15. The number of units of area in a rectangle is equal to the product of the number of units of length in one side multiplied by the number of units of length in the other.

Or, more shortly : Area of rectangle = length $\times$ breadth.

Prop. 16. The area of a triangle = $\frac{1}{2}$ base $\times$ altitude.

The Concurrencies of the Triangle.

Prop. 17. The lines bisecting the sides of a triangle at right angles are concurrent (i. e. meet at a point).

The point in which they meet is the centre of the circle passing through the three vertices and is called the **circumcentre**.

Prop. 18. The lines drawn from the vertices to bisect the opposite sides are concurrent.

These lines are called the **medians** and the point of concurrency is called the **centroid**.

Prop. 19. (a) The lines bisecting the angles are concurrent.

The point of concurrency is the centre of the circle that touches all the sides, and is called the **incentre**.

(b) If two of the sides be produced, the lines bisecting the exterior angles so formed and the line bisecting the interior angle contained by the produced sides are concurrent.

The point of concurrency is the centre of the circle that touches the two sides when produced and the third side (not produced); it is called an **e-centre**.

Prop. 20. The perpendiculars let fall from the vertices on the opposite sides are concurrent.

The point of concurrency is called the **orthocentre**.

The Circle.

Prop. 21. The straight line passing through the centre, at right angles to a chord, bisects the chord.

Prop. 22. (a) The angle at the centre of a circle is twice the angle at the circumference on the same arc. (b) Angles in the same segment are equal. (c) The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

Prop. 23. The tangent at any point is at right angles to the radius drawn to that point.

Prop. 24. (a) Two tangents can be drawn to a circle from any external point. (b) The parts of these tangents between the external point and the points of contact are equal. (c) The line joining the external point to the centre bisects the angle between the tangents.

Prop. 25. The ratio of the circumference of any circle to its diameter is the same for all circles.

This ratio is denoted by the symbol π ; its value is 3.1416 correct to five significant figures.

Prop. 26. The area of a circle equals the area of the rectangle contained by the radius and a straight line equal to half the circumference

This is usually expressed in the formula : $\text{Area} = \pi r^2$.

GRAPHS*

Geometrical. If two straight lines are drawn in a plane, the position of any point in the plane can be determined by means of its distances from those lines.

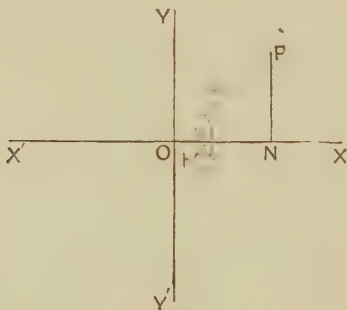


Fig. 1.

It is usual to draw one of the lines horizontal and the other perpendicular to it. The customary notation is shown in Fig. 1.

$X'OX$ is called the **axis of x** ;

$Y'OY$ is called the **axis of y** ;

O is called the **origin**;

ON is called the **abscissa** of the point P ;

NP is called the **ordinate** of the point P .

The abscissa and ordinate are called the **co-ordinates**† of the point P .

* For a fuller treatment of Graphs see *School Algebra* published by the Clarendon Press.

† These co-ordinates are called Cartesian co-ordinates because they were first used by the French mathematician, Descartes.

The abscissa is said to be positive if drawn to the right, negative if drawn to the left. Similarly, the ordinate is positive if drawn upwards from N , negative if drawn downwards. The number of units of length in ON , preceded by the proper sign, is usually denoted by x , and the number of units of length in NP , preceded by the proper sign, is denoted by y . In each case the sign $+$ is often omitted.

Thus, in Fig. 2, the co-ordinates of A are $x = -4$, $y = 2$, of C , $x = 4$, $y = 3$, of L , $x = 0$, $y = -7$.

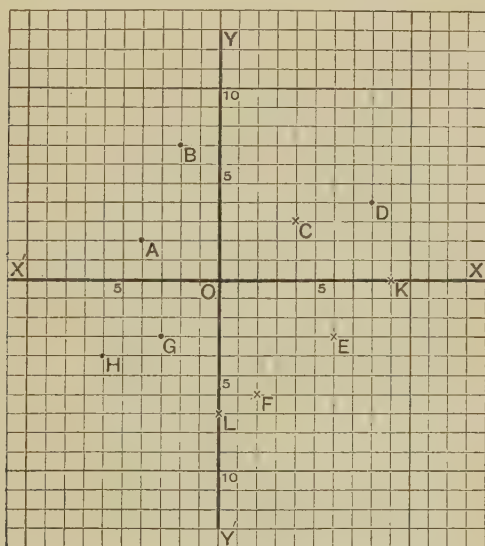


Fig. 2.

Very often a point is described by writing the values of the co-ordinates in brackets; e.g. the point H might be described as the point $(-6, -4)$.

Exercise. Write down the co-ordinates of all the points in Fig. 2.

Graphs of Statistics. The magnitude of any quantity may be represented by a straight line which contains as many units of length as the quantity contains units of its own kind.

If two quantities are changing their values at the same time,

the simultaneous values may be represented in the same figure by taking horizontal lengths to represent one magnitude and vertical lengths to represent the other.

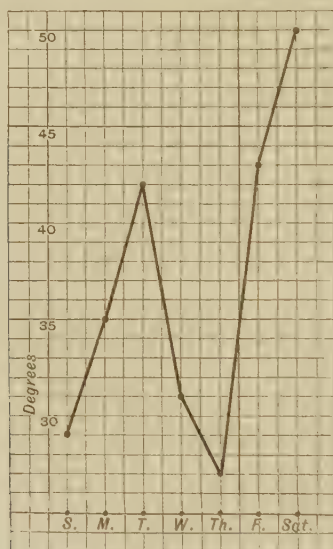


Fig. 3.

In Fig. 3 the changing quantities are time and temperature; and the dots show that at noon on Sunday the temperature was 29° , on Monday the temperature was 35° , &c. In fact the figure conveys the same information as the following table:—

	Sun.	Mon.	Tues.	Wed.	Thurs.	Fri.	Sat.
Temp.	29°	35°	42°	31°	27°	43°	50°

If there is no information about intermediate temperatures, the points are joined by a series of straight lines. The figure now forms a graph.

In describing such a graph we should say that the abscissae represent time and the ordinates temperature.

Graphs of functions. If two quantities x and y are such that a change of value in the one causes a change of value in the other, then either of them is said to be a *function* of the other.

This is expressed thus: $y = f(x)$, or $x = f(y)$ where $f(x)$ means a function of x . A graph can be drawn in which the abscissae are proportional to the values of x and the ordinates to the values of y . This graph is called the graph of the function $f(x)$ or of the equation $y = f(x)$. This may be more easily understood by considering a few algebraical functions.

Example I. Draw the graph when $y = \frac{3}{5}x - \frac{2}{5}$.

(Choose values of x which will make y a whole number.)

$$y = \frac{1}{5}(3x - 2)$$

x	- 6	- 1	4	9
$3x - 2$	- 20	- 5	10	25
y	- 4	- 1	2	5

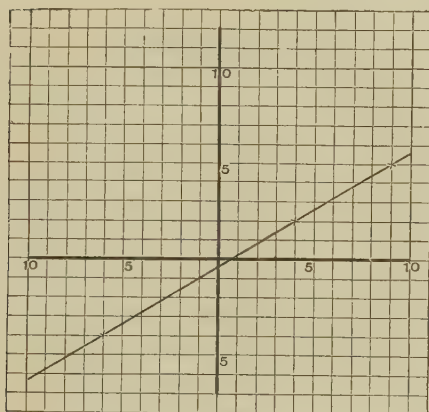


Fig. 4.

When the points corresponding to these values of x and y are plotted, it is found that they lie on the straight line shown in Fig. 4. It is also found

(i) That any simultaneous values of x and y connected by the given equation are the co-ordinates of some point on this straight line;

(ii) That the co-ordinates of any point on the straight line satisfy the equation.

It is found by experience (and can be proved from the geo-

metrical propositions on proportion) that, when x and y are connected by an equation of the first degree, the graph is always a straight line.

Example II. In the same figure draw the graphs of

$$y = x^2 - 3x + 2 \text{ and } x = 2y^2 + 3.$$

Neither of these equations is of the first degree, therefore neither of the graphs is a straight line. At least six points must be found on each.

$$y = x^2 - 3x + 2.$$

(i)

x	-3	-2	-1	0	1	2	3	4	5
x^2	9	4	1	0	1	4	9	16	25
$-3x$	9	6	3	0	-3	-6	-9	-12	-15
y	20	12	6	2	0	0	2	6	12

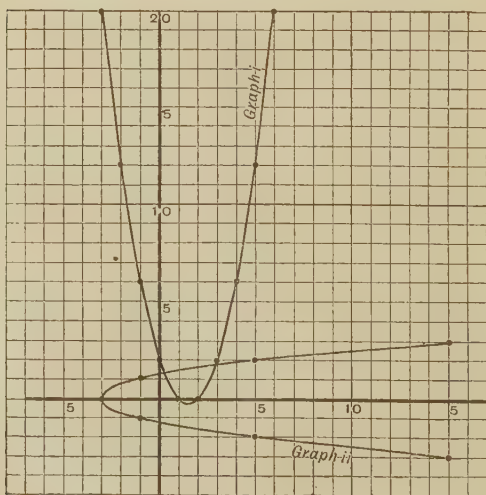


Fig. 5.

$$x = 2y^2 - 3.$$

(ii)

y	-3	-2	-1	0	1	2	3	4
$2y^2$	18	8	2	0	2	8	18	32
x	15	5	-1	-3	-1	5	15	29

The co-ordinates of every point on graph i satisfy the first equation, and of every point on graph ii satisfy the second equation. Hence the co-ordinates of any points which are on both graphs, that is, the co-ordinates of the points of intersection, satisfy both equations.

Fig. 5 shows, therefore, that the values $x = \cdot 3$, $y = 1\cdot 3$, and $x = 2\cdot 8$, $y = 1\cdot 6$, are the solutions of the two equations.

This graphical method of solving equations is very useful, but is, of course, only approximate. If more accurate answers are required, the graphs must be drawn on a larger scale in the neighbourhood of their points of intersection.

LOGARITHMS *

Fractional and Negative Indices. It is shown in Algebra that

$$x^{\frac{p}{q}} = q\sqrt[q]{x^p}, \quad x^0 = 1, \quad x^{-m} = \frac{1}{x^m},$$

where p and q are any positive integers and x is any positive quantity, integral or fractional.

A fractional index may be expressed as a decimal; thus such expressions as $4\cdot^{36}$, $10\cdot^{201}$ have a definite value. This value could in theory be found by reducing the decimal to a vulgar fraction and then replacing the power with a fractional index by a root, e.g.

$$10\cdot^{201} = 10^{\frac{201}{1000}} = 1000\sqrt[1000]{10^{201}}.$$

This is obviously not practical. The value can be found by a graphical method which is easy but only approximate.

Draw the graph of $x = 10^y$.

* For a fuller treatment of Indices and Logarithms see *School Algebra*, Chapters XXI and XXII

y	0	·5	·25	·125	·75	·625	·875	1
x	1	3·16	1·76	1·33	5·62	4·26	7·49	10

The values of x are obtained as follows:—

$10^0 = 1$ by definition given above.

$10^{·5} = 10^{\frac{1}{2}} = \sqrt{10}$.

$10^{·25} = 10^{\frac{1}{4}} = \sqrt{\sqrt{10}}$; similarly $10^{·125} = \sqrt{10^{·25}}$.

$10^{·75} = 10^{·5+·25} = 10^{·5} \times 10^{·25}$, &c.

The graph is shown on a small scale in Fig. 6.

It is seen that $10^{·30}$ is almost exactly 2; any other power of 10 can be found approximately from this graph when the index is between 0 and 10.

Definition of a logarithm. The logarithm of a number to a given base is the index of the power to which the base must be raised to equal the number. Thus $3^2 = 9$, therefore the logarithm of 9 to base 3 equals 2; this is written $\log_3 9 = 2$.

In dealing with numbers the base is 10. In the remainder of this chapter it is assumed that the base is always 10, so that $\log 731$ means logarithm of 731 to base 10.

The equation $x = 10^y$ may be written $y = \log x$.

Hence Fig. 6 provides an approximate means of finding the logarithm of any number between 1 and 10.

Characteristic and mantissa. Consider a number, such as 4378. It means

$$4 \times 10^3 + 3 \times 10^2 + 7 \times 10 + 8.$$

Also a decimal number, such as ·0376, means

$$\frac{0}{10} + \frac{3}{10^2} + \frac{7}{10^3} + \frac{6}{10^4}.$$

If we use negative indices, this may be written

$$·0376 = 3 \times 10^{-2} + 7 \times 10^{-3} + 6 \times 10^{-4}.$$

Similarly

$$537·13 = 5 \times 10^2 + 3 \times 10^1 + 7 \times 10^0 + 1 \times 10^{-1} + 3 \times 10^{-2}.$$

It follows that

$$4378 > 10^3 \text{ but } < 10^4, \therefore \log 4378 = 3 + \text{a decimal};$$

$$·0376 > 10^{-2} \text{ but } < 10^{-1}, \therefore \log ·0376 = -2 + \text{a decimal};$$

$$537·13 > 10^2 \text{ but } < 10^3, \therefore \log 537·13 = 2 + \text{a decimal};$$

$$4·37 > 10^0 \text{ but } < 10^1, \therefore \log 4·37 = 0 + \text{a decimal}.$$

We now see (i) that the logarithm of any number consists of an integer (which may be positive, zero, or negative) and a positive

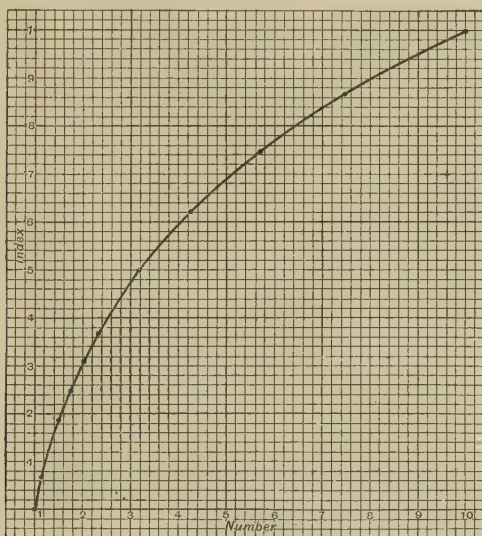


Fig. 6.

decimal, (ii) that the integer is the index of the highest power of 10 contained in the number.

The integral part of a logarithm is called the **characteristic**; the decimal part the **mantissa**.

Index	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
Number	3	7	8	9	0	0						
"			4	3	7	6	5	2				
"						3	4	6	7	8		
"						0	0	0	7	3	4	8
"			2	0	0	0	0	0	0			

Consideration of the preceding table shows that the characteristic (i.e. the highest index) may always be found by the following rule: Count from the unit place to the first significant figure (i.e. the first figure which is not 0), the unit place being counted as nothing. The characteristic is positive or zero if the number is greater than one, negative if it is less than one.

The mantissa is independent of the position of the decimal point. An example will make this clear.

Given that $\log 4.376 = .6411$, find $\log 4376$ and $\log .004376$.

$$4376 = 1000 \times 4.376$$

$$\text{but } 4.376 = 10^{.6411} \text{ since } \log 4.376 = .6411.$$

$$\therefore 4376 = 10^3 \times 10^{.6411} = 10^{3.6411}, \text{ i.e. } \log 4376 = 3.6411$$

$$.004376 = \frac{1}{1000} \times 4.376 = 10^{-3} \times 10^{.6411} = 10^{-3+.6411}$$

$$\therefore \log .004376 = -3 + .6411.$$

The negative sign of a characteristic is always placed on top and the + before the decimal is omitted. Thus $\log .004376 = \bar{3}.6411$.

To find the logarithm of any number.

(a) Four-figure tables. The mantissa is found from tables, of which a specimen is given below.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7

Consider the logarithms of 5467 and .05467.

By counting, the characteristic of $\log 5467$ is found to be 3, and that of $\log .05467$ to be $\bar{2}$.

Both logarithms have the same mantissa. Look for 54 in the extreme left-hand column. In the same line with 54 and under 6 we find 7372; this is the mantissa of $\log 546$. Under the 7 in the small columns to the right, and in a line with 54, we find 6; this must be added to the last digit of the mantissa already found. Hence the mantissa is .7378.

Therefore $\log 5467 = 3.7378$ and $\log .05467 = \bar{2}.7378$.

(b) Five-figure tables. Find the logarithm of 346·73.

Proceeding as with four-figure tables, we find that the mantissa of log 346 is ·53908. Under 7 in the side columns, we find 88; this must be added to the last two digits already found. For a 3 in the fourth place we should add 38, for a 3 in the fifth place we add, therefore, $\frac{1}{10} \times 38$, i. e. 4 to nearest integer. Hence

$$\log 3467\cdot3 = 2\cdot53908 + 88 + 4 = 2\cdot54000.$$

To find the number corresponding to any logarithm.

Method I. Reverse the process for finding a logarithm. Suppose the logarithm is 3·7271.

Look in the logarithms for the mantissa nearest to 7271, but less than it. We find 7267, level with 53 and under 3; the first three figures of the number are 533. This leaves $7271 - 7267 = 4$; in the right-hand columns 4 is found under 5. Hence the first four figures are 5335.

The characteristic is 3, therefore the left-hand digit 5 represents 5×10^3 ; hence the number is 5335.

The number is called the **antilogarithm** of the logarithm.

Method II. If tables of antilogarithms are available, they are used in the same way as logarithm tables.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12

Look for ·72 in the left-hand column; level with ·72 and under 7 we find 5333; in the small columns we find 1 under 1. Hence the first four figures are 5334.

The characteristic is 3; as before, the number is 5334.

Note.—The two methods give results differing by 1 in the last figure; this shows that the number is between the two results. On using five-figure tables, it is found that the antilogarithm of 3·7271 is 5334·5.

Exercises. Find the value of

- (1) $\sqrt{319 \cdot 2} \times 1 \cdot 756$. Ans. 31.37.
 (2) $\cdot 03056 \times 0 \cdot 4105$. Ans. $\cdot 01254$.
 (3) $3 \cdot 142 \times (71 \cdot 43)^2$. Ans. 16030
 (4) $\frac{4}{3} \times 3 \cdot 142 \times (9 \cdot 67)^3$. Ans. 3787.
 (5) $254 \cdot 3 \div 0 \cdot 09027$. Ans. 2817.
 (6) $\frac{(\cdot 1136)^{\frac{2}{3}} \times \sqrt{81 \cdot 86}}{\frac{2}{3} \times \sqrt[3]{2000}}$. Ans. $\cdot 1874$.

$$2 \sin \theta = 2 - \frac{11}{29}$$

$$= \frac{40}{29}$$

THE GREEK ALPHABET

Greek letters are used so frequently in Trigonometry and other branches of Mathematics that it is useful to have the complete alphabet for reference.

Nome.	Small.	Capital.
alpha	α	A
beta	β	B
gamma	γ	Γ
delta	δ	Δ D
epsilon	ϵ	E
zēta	ζ	Z
ēta	η	H
thēta	θ	Θ
iōta	ι	I
kappa	κ	K
lambda	λ	Λ
mu	μ	M
nu	ν	N
xi	ξ	Ξ
omicron	$\omicron$	O
pi	π	Π
rho	ρ	P
sigma	σ	Σ
tau	τ	T
upsilon	υ	Υ
phi	ϕ	Φ
chi	χ	X
psi	ψ	Ψ
omega	ω	Ω

CHAPTER I

ANGLES AND THEIR MEASUREMENT

1. Any angle such as BAC (Fig. I) may be thought of as having been formed by rotating the line AC about the point A from the position of coincidence with AB to its final position AC .

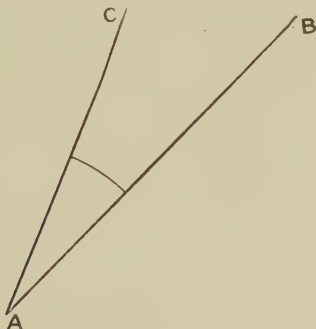


Fig. I.

This way of regarding an angle shows clearly the intimate connexion between angles and arcs of circles and this connexion leads to the usual method of measuring angles.

2. The Degree.* From very early times it has been the custom to divide the circumference of a circle into 360 equal parts or

* 'The current sexagesimal division of angles is derived from the Babylonians through the Greeks. The Babylonian unit angle was the angle of an equilateral triangle; following their usual practice this was divided into 60 equal parts or degrees, a degree was subdivided into 60 equal parts or minutes, and so on; it is said that 60 was assumed as the base of the system in order that the number of degrees corresponding to the circumference of a circle should be the same as the number of days in a year which it is alleged was taken, at any rate in practice, to be 360.' (From *A Short Account of the History of Mathematics*, by W. W. Rouse Ball.)

degrees, each degree into 60 parts or minutes,* each minute into 60 seconds.¹

The angle at the centre of a circle, subtended by an arc of 1 degree, is taken as the unit angle, and it, too, is called a degree; it is divided into minutes and seconds in the same way as the arc degree.

The notation used is shown in the following example:— $47^{\circ} 15' 37''$ is read 47 degrees 15 minutes 37 seconds.

If the line makes a complete rotation, thus returning to its original position, it has turned through an angle of 360° .

A right angle is produced by one-quarter of a complete rotation, and is, therefore, equal to 90° .

If two angles together equal a right angle, either of them is called the **complement** of the other. When the sum equals two right angles, either angle is the **supplement** of the other.

3. Positive and Negative Angles. In discussing the properties of a single angle it is usual to draw the initial line so that it is horizontal and to name it OA . If the rotating line moves in a direction opposite to that of the hands of a clock, the angle is said to be **positive**; if in the same direction as the hands of a clock, the angle is **negative**.

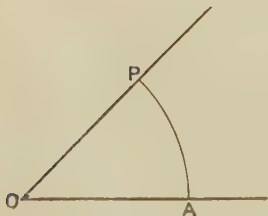


Fig. II.

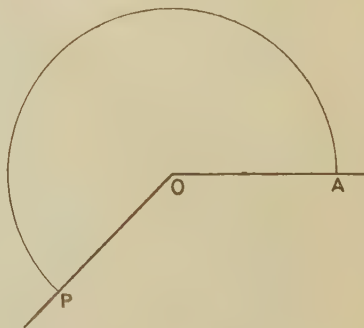


Fig. III.

* Minutes derived from the Latin *partes minutae*; seconds from the Latin *partes minutae secundae*.

In Fig. II the line OP has made $\frac{1}{8}$ of a complete turn, hence the angle $AOP = \frac{1}{8} \times 360^\circ = 45^\circ$; in Fig. III. the angle AOP is reflex* and is equal to $\frac{5}{8} \times 360^\circ = 225^\circ$. If, in Fig. II, the line OP reached its position by turning in the negative direction it would have made $\frac{7}{8}$ of a complete turn so that the reflex angle AOP in Fig. II = -315° . Similarly in Fig. III the obtuse angle $AOP = -135^\circ$.

4. Angles unlimited in size. In Fig. II the line OP might have made one, two, or any number of complete turns, either positive or negative, and then have moved on to its final position: hence the angle AOP may represent 405° , or 765° , or -675° . All possible values are included in the general formula

$$AOP = 360n + 45,$$

where n is any whole number, positive, zero, or negative.

Unless the problem under discussion allows the possibility of the angle being greater than 360° , it is always assumed that the angle is less than 360° .

5. The Grade. When the metric system was invented, the French Mathematicians introduced a new unit, the Grade, such that

$$100 \text{ grades} = 1 \text{ right angle},$$

$$100 \text{ minutes} = 1 \text{ grade},$$

$$100 \text{ seconds} = 1 \text{ minute}.$$

This system never came into general use, even in France, and now exists only in old-fashioned examination papers.

Examples Ia.

1. Find the complement of each of the following angles: 32° , $47^\circ 23'$, $75^\circ 13' 14''$, $68^\circ 0' 13''$, $27^\circ 42' 18.6''$.

2. Write down the supplements of 75° , $68^\circ 14'$, $115^\circ 17' 48''$, 90° , $78^\circ 24' 36''$.

3. The angles of a triangle are found to be $42^\circ 13' 17''$, $73^\circ 47' 5''$, $64^\circ 0' 38''$. Is this correct?

4. Two angles of a triangle are $17^\circ 43'$, $92^\circ 16'$; calculate the third angle.

5. In a triangle ABC , $\frac{1}{2}(A + B) = 77^\circ 29'$ and $\frac{1}{2}(A - B) = 16^\circ 25'$; find all the angles.

* A reflex angle is an angle greater than two right angles, but less than four right angles.

6. Express in degrees, minutes, and seconds the angle of (a) a square, (b) a regular pentagon, (c) a regular heptagon.

7. Express each of the angles of question 6 in grades.

8. The magnitude of an angle may be expressed either as D degrees or G grades; find the equation connecting D and G .

9. Draw the angles A and $\frac{1}{3}A$ in each of the following cases: (a) $A = 54^\circ$, (b) $A = 414^\circ$, (c) $A = 774^\circ$, (d) $A = 1134^\circ$, (e) 234° , (f) -126° .

10. Through what angles do the hour, minute, and second hands of a watch respectively turn between $12^h 30'$ a.m. and $5^h 3'$ a.m.?

6. The ratio of the length of the circumference of a circle to the length of its diameter is the same for all circles. This constant value is denoted by the Greek letter π (pronounced *pi*), so that if

the circumference = c units of length,
and diameter = d units of length,

$$\text{then} \quad \frac{c}{d} = \pi.$$

The value of π can be found, correct to two or three significant figures, by actual measurement. By geometrical and trigonometrical calculations its value can be calculated to any desired number of places.

Correct to 5 significant figures, $\pi = 3.1416$.

Correct to 6 significant figures, $\pi = 3.14159$.

For mental calculations π may be taken as $3\frac{1}{7}$.

7. By using Prop. 5, p. 9, problems dealing with the lengths of circular arcs may often be solved.

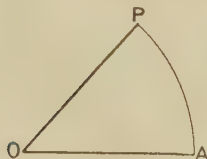


Fig. IV.

Example. Find the length of an arc which subtends an angle of 49° at the centre of a circle whose radius is 5 feet.

$$\frac{\text{arc } AP}{\text{semi-circumference}} = \frac{\text{angle } AOP}{2 \text{ right angles}},$$

$$\text{i. e.} \quad \frac{\text{arc } AP}{5\pi} = \frac{49^\circ}{180^\circ}.$$

The calculation is easily completed.

Similarly $\frac{\text{area of sector } AOP}{\text{area of circle}} = \frac{\text{angle } AOP}{4 \text{ right angles}},$ (Prop. 26) /

i. e. $\frac{\text{area of sector } AOP}{25\pi} = \frac{49}{360}.$ V

8. Circular measure. By the method of the last section it is easily shown that the length of an arc of a circle, radius r , subtending an angle A° at the centre is $r \frac{A\pi}{180}$.

In many other formulae the fraction $\frac{A\pi}{180}$ occurs in connexion with the angle A° . In theoretical work it has, therefore, been found convenient to use another unit angle, which simplifies formulae considerably.

The **radian** is the angle subtended at the centre of any circle by an arc equal in length to the radius.

Let x° equal 1 radian

$$\frac{\text{arc equal to radius}}{\text{semi-circumference}} = \frac{\text{angle of 1 radian}}{2 \text{ right angles}} = \frac{x^\circ}{180^\circ},$$

i. e. $\frac{r}{\pi r} = \frac{x^\circ}{180^\circ};$

$\therefore x^\circ = \frac{180}{\pi}.$

Since π is the same for all circles, it follows that the radian is the same for all circles and may, therefore, be taken as a unit of measurement.

The number of radians in an angle is often called the **circular measure** of the angle. For this reason the symbol ^{c} is used to show that the angle is measured in radians, e. g. 2^c means 2 radians.

When the radian is the unit angle, it is customary to use Greek letters to denote the number of radians, and the symbol ^{c} is then often omitted. When capital English letters are used, it is usually understood that the angle is measured in degrees.

Examples I b.

1. How many times is an arc equal to the radius contained in the semi-circumference? Reduce 180° , 90° , 60° , 30° to radians. (Do not substitute for π .)

2. Show by simple geometry that the radian is less than 60° .

3. How many radians are there in 10° , 75° , 138° , respectively? Give the answers correct to 2 decimal places.

4. Express the angle of (i) an isosceles right-angled triangle, (ii) a regular nonagon, in circular measure. Give the answers in terms of π .

5. One angle of a triangle is $\frac{1}{3}\pi$, another is $\frac{1}{4}\pi$; what is the circular measure of the third angle?

6. Find the length of an arc of a circle which subtends an angle 78° at the centre, the radius being 18 feet.

7. An arc of length 5 feet subtends an angle of 132° at the centre; what is the radius of the circle?

8. Find the area of the sector of a circle if the radius is 12 feet and the angle 40° .

9. What time does the minute hand of a watch take to turn through (i) 3000° , (ii) 3000 grades, (iii) 3000 radians?

10. Fill in the missing values in the following table, which gives data about circular arcs.

	Radius.	Angle.	Length.
(1)	5 inches	2 radians	
(2)	7.6 centimetres	74.6 grades	
(3)		314 degrees	418 feet
(4)	100 yards	radians	220 yards
(5)	320 metres	degrees	1 kilometre
(6)	yards	5 radians	half a mile

11. Express in radians the angle of a sector of a circle, being given that the radius is 7 inches and the area of the sector 100 sq. inches.

12. Show that the length of an arc subtending an angle θ° at the centre of a circle, radius r , is $r\theta$. What is the area of the corresponding sector?

13. Find the circular measure of $1'$ and of $1''$, correct to 5 significant figures.

The angle between two consecutive points of the compass is also called a *point*, thus N. 2 points E. is the same as NNE.; WSW. $\frac{1}{2}$ W. means $\frac{1}{2}$ a point W. of WSW.

The ordinary degree is sometimes used in defining a direction, for instance ENE. can be referred to as $22\frac{1}{2}^\circ$ N. of E. Similarly we may have 32° W. of N., 40° S. of W., &c.

10. Latitude and Longitude. The position of a point on a sphere can be defined by two angles, which may be compared with the abscissa and ordinate of plane geometry. These angles are easily understood by considering the special case of Longitude and Latitude.

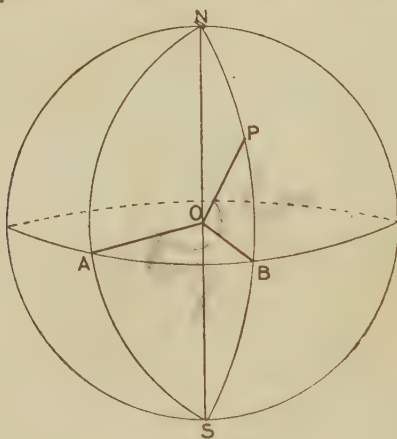


Fig. V.

In Fig. V the meridian through Greenwich cuts the equator at A; the meridian through P cuts at B. O is the centre of the Earth.

The **Longitude** of P is the angle AOB and may be either East or West of the Greenwich meridian.

The **Latitude** of P is the angle POB and may be either North or South of the Equator.

Note.—A geographical or nautical mile is the length of an arc of a meridian (or of the equator) subtending an angle of $1'$ at the centre of the earth.

A ship travelling at the rate of 1 nautical mile per hour is said to have a speed of one knot.

11. Gradient. It is usual to estimate the inclination to the horizontal of a road or hill by the distance risen vertically for a certain horizontal distance. Thus a hill might be said to rise 3 in 5; this would mean that if a horizontal line were drawn through a point B on the hill to meet the vertical line through a lower point A at C , then AC/BC would equal $\frac{3}{5}$. The hill is said to have a gradient or slope of 3 in 5.

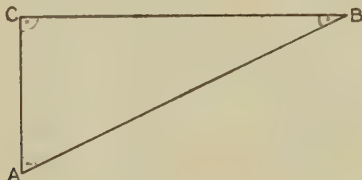


Fig. VI.

It is clear that in many cases it is easier to measure AB than BC ; and some books take a gradient of 3 in 5 to mean a rise of 3 vertically for a distance 5 measured along the incline; so that in the figure AC/AB would be $\frac{3}{5}$. This latter interpretation of gradient is very common in books on Theoretical Mechanics.

If the inclination is small, it makes no practical difference which interpretation of gradient is taken.

It should be noticed that the angle is the same whatever units be used; that is whether we consider a rise of 3 inches in 5 inches, 3 furlongs in 5 furlongs, 3 miles in 5 miles. This follows from Prop. 12 c.

Examples I c.

1. Express in degrees the angle between

- | | |
|-----------------------------|--|
| (a) NNE. and E. by N.; | (f) S. 2 points W. and W. 2 points S.; |
| (b) W. by S. and SE. by N.; | (g) 40° N. of W. and 30° E. of S.; |
| (c) ESE. and NE. by N.; | (h) NE. by E. and 1 point W. of N.; |
| (d) NNW. and SSE.; | (i) 30° S. of W. and ESE.; |
| (e) N. by W. and SW.; | (k) S. 2 points W. and W. 2 points N. |

In the following questions take the radius of the Earth to be 4000 miles.

2. Two places on the Equator are 300 miles apart, find the difference of their Longitudes.

3. Quito (Longitude 79° W.) and Macapa (Longitude $51\frac{1}{2}^\circ$ W.) are both on the Equator, find the distance between them. What time is it at Macapa when it is noon at Quito ?

4. Find the distance between Poole (Lat. $50^\circ 43'$ N., Long. $1^\circ 59'$ W.) and Berwick (Lat. $55^\circ 46'$ N., Long. $1^\circ 59'$ W.).

5. Find the distance between Cape Breton Island (Lat. $45^\circ 50'$ N., Long. 60° W.) and the Falkland Isles (Lat. $51^\circ 32'$ S., Long. 60° W.).

Oral Questions.

1. What is a degree ? How many degrees are there in the angle of a regular pentagon ?

2. How big is each acute angle of an isosceles right-angled triangle ?

3. One angle of a triangle is 4° , another 30° , how big is the third angle ?

4. What is meant by a negative angle ? When screwing an ordinary screw in, is the turning in the positive or negative direction ?

5. Does the earth rotate in the positive or negative direction ? In which direction does the sun appear to move ?

6. Do you usually draw a circle in the positive or negative direction ?

7. The needle of a mariner's compass is deflected from its normal position through a positive angle $33\frac{1}{4}$ degrees, to what point of the compass does it then point ?

8. Express the following angles in circular measure : 90° , 60° , 180° , 45° , 30° . (Give the answers in terms of π .)

9. What is the locus of all places having latitude 35° N. ?

10. What is the locus of all places having longitude 15° W. ?

11. It is noon at the same time at two different places, what do you know about their longitudes or latitudes ?

12. Give the latitude and longitude of the N. pole.

Examples I.

1. In a triangle ABC , $A = 43^\circ 15'$, $B = 67^\circ 38'$, calculate the number of degrees in (i) the angle C , (ii) the angle subtended at the centre of the circumcircle by the side BC , (iii) the angle subtended at the centre of the inscribed circle by the side BC .

2. Express in circular measure, correct to 3 significant figures, (a) the supplement of 1.37 radians, (b) 74° , (c) the angle of a regular octagon.

3. Define a radian and a grade. If an angle, containing D degrees, may be expressed as either θ radians or G grades, prove that $D/180 = \theta/\pi = G/200$.

4. The hands of a clock are coincident at noon, through what angle does the hour hand turn before they next coincide?

5. Prove that whatever be the radius of a circle the size of the angle at the centre, which subtends an arc equal to the radius, is constant. What is this angle called? Show, by a geometrical construction, that it is a little less than 60° .

6. A wheel of a cart is 4 feet in diameter, through what angle does it turn when the cart moves forward 10 feet?

7. Explain how to find the length of a circular arc being given the number of degrees in the angle subtended at the centre and the length of the radius.

8. Two places on the Equator differ in longitude by $37^\circ 16'$, find the distance between them, correct to three significant figures. (Radius = 4000 miles.)

9. Find the distance between a place, longitude $45^\circ 17'$ E., latitude 0° , and another place, longitude $38^\circ 43'$ W., latitude 0° .

10. Through what angle does the Earth turn between 9.30 a.m. and 4 p.m.?

11. When it is noon at Greenwich what time is it at (a) Calcutta ($88^\circ 15'$ E.), (b) New York (74° W.), (c) Hawaii (156° W.)?

12. The co-ordinates of two points P and Q are (7, 8), (9, 11) respectively, find the gradient of the line PQ .

13. Draw an angle $AOP = 35^\circ$, in OP take 3 points P , Q , R such that $OP = 1$ inch, $OQ = 1.7$ inch, $OR = 2.3$ inches. From P draw PH at right angles to OA , at Q draw QK at right angles to OQ , and from R let fall RL perpendicular to OA . Measure OH , OK , OL and calculate, to 3 decimal places, the ratios OH/OP , OQ/OK , OL/OR . Justify the result.

14. Explain what is meant by a radian, and find how many degrees and minutes it contains.

Express in degrees, and also in radians, the angle of a regular polygon of 100 sides.

15. An explorer reaches a latitude of $87^{\circ} 28' 48''$. Find how many miles he is distant from the pole, assuming the earth to be a sphere whose circumference is 25000 miles.

16. Find the gradient of a straight line joining two points whose co-ordinates are (x', y') and (x'', y'') . Hence find the equation of the straight line.

CHAPTER II

THE TRIGONOMETRICAL RATIOS

12. Definitions. Let OA the initial line be taken as axis of x , the axis of y being perpendicular to it at O ; in the final

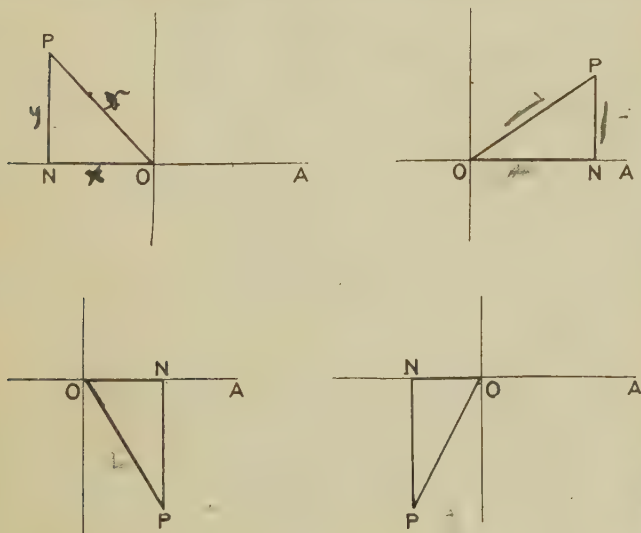


Fig. VII.

position of the rotating line take any point P . Figure VII shows four possible types of positions of OP .

Let fall PN perpendicular to OA or OA produced, so that ON is the abscissa of P and PN the ordinate. Then

$$\text{sine of } \angle AOP = \frac{NP}{OP} = \frac{\text{ordinate}}{\text{radius}} = \frac{y}{r},$$

$$\text{cosine of } \angle AOP = \frac{ON}{OP} = \frac{\text{abscissa}}{\text{radius}} = \frac{x}{r},$$

$$\text{tangent of } \angle AOP = \frac{NP}{ON} = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}.$$

These are the most important ratios; the others are their reciprocals, viz.:

$$\text{cosecant of } \angle AOP = \frac{OP}{NP} = \frac{\text{radius}}{\text{ordinate}} = \frac{r}{y},$$

$$\text{secant of } \angle AOP = \frac{OP}{ON} = \frac{\text{radius}}{\text{abscissa}} = \frac{r}{x},$$

$$\text{cotangent of } \angle AOP = \frac{ON}{NP} = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}.$$

The following abbreviations are usually used:

$\sin A$ instead of sine of $\angle AOP$,
 $\cos A$ „ „ cosine of $\angle AOP$,
 $\tan A$ „ „ tangent of $\angle AOP$,
 $\text{cosec } A$ „ „ cosecant of $\angle AOP$,
 $\sec A$ „ „ secant of $\angle AOP$,
 $\cotan A$ „ „ cotangent of $\angle AOP$.

Similarly, if $\angle AOP$ is measured in radians, $\sin \theta$, $\cos \phi$, $\text{cosec } \psi$, &c., are used.

13. Trigonometry was developed by Arabian and Greek astronomers who based their work on the circular arc and not on the angle. In the Middle Ages this early mathematical work was translated into Latin, and so the present names of the ratios were derived. The following section shows the reasons for these names.

14. Draw a circle with centre O cutting the initial line at A and the perpendicular to it at B .

Take a point P on the circumference of the circle.

Draw the tangent at A and produce OP to meet it at T .

Draw PN perpendicular to OA .

AT was called the **tangent** of the arc AP .

OT , which cuts the circle, was called the **secant** of the arc AP .

NP was called the **sine** * of the arc AP .

Clearly the lengths of AT , OT , NP change when the radius OA changes, even if the angle AOP remain constant.

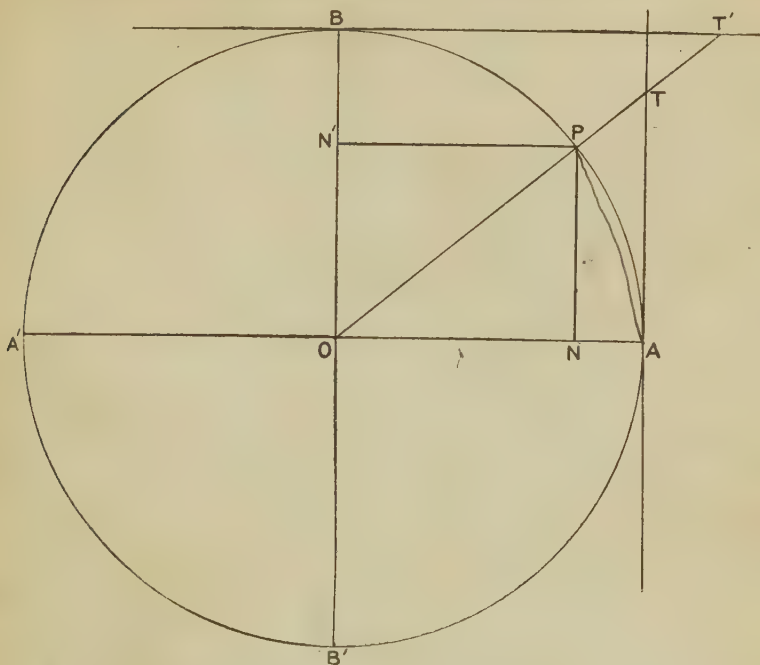


Fig. VIII.

But from similar triangles it is seen that, if the angle is constant, the ratios of AT , OT , NP to the radius are also constant. Hence, as Trigonometry developed, it was seen to be advisable to divide by

* The word 'sine' is derived from the Latin *sinus*. If in Fig. VIII PN be produced to meet the circumference at P' , then PAP' resembles a bow (Latin *arcus*) of which PNP' is the string or chord (Latin *chorda*). To use the bow, the string is pulled till N touches the bosom (Latin *sinus*); hence PN is called the sine. NA is often called the sagitta of the arc.

the radius and to treat the subject as depending on the angle AOP rather than on the arc AP . Thus we have

$$\text{angle } AOP = \frac{\text{arc } AP}{\text{radius}} \quad (\text{when the angle is measured in radians}),$$

$$\sin AOP = \frac{\text{sine of arc } AP}{\text{radius}} = \frac{NP}{OP},$$

$$\tan AOP = \frac{\text{tangent of arc } AP}{\text{radius}} = \frac{AT}{OA} = \frac{NP}{ON},$$

$$\sec AOP = \frac{\text{secant of arc } AP}{\text{radius}} = \frac{OT}{OA} = \frac{OP}{ON}.$$

Now make a similar construction for the complementary arc BP . Then

$$\begin{aligned} \text{sine of the complement of the angle } AOP &= \frac{\text{sine of the complementary arc } BP}{\text{radius}} \\ &= \frac{N'P}{OP} \\ &= \frac{ON}{OP}. \end{aligned}$$

'Sine of the complement of' was shortened into co-sine. Possibly 'complementary sine' was an intermediate stage. Similarly, co-tangent and cosecant were derived.

Since the values of the ratios depend on the values of the angle, the term **Trigonometrical Functions** is often used instead of **Trigonometrical Ratios**. Frequently the ratios are referred to as **Circular Functions**.

15. Ratios rarely used. In Fig. VIII

NA is called the versine (i. e. versed sine) of the arc AP .

$N'B$ is called the coversine of the arc AP .

AP (not joined in the figure) is the chord of the arc AP .

If we divide each of these by the radius we get the corresponding ratios of the angles AOP . These ratios are very rarely used. Another function that is now rarely used is the haversine, i. e. half the versed sine.

16. Projection Formulae. It is useful to remember that

ON (i. e. the projection of the radius on the initial line)

$$= r \cos \theta$$

and NP (i. e. the projection of the radius on a line perpendicular to the initial line)

$$= r \sin \theta.$$

17. Polar co-ordinates. The position of a point P is determined if the distance OP from a fixed point is known and also the angle this distance makes with a fixed line OX through O . The length is usually denoted by r and the angle by θ ; these are the polar co-ordinates of P . In this connexion O is called the pole.

18. Graphs of Trigonometrical Functions.

The definitions of the last section lead to an easy method of drawing the graphs. On page 42 the sine graph (i.e. the graph of the equation $y = \sin x$) is given. It is obtained as follows:

Step 1. On the extreme left of the paper (which should be ruled in squares) draw a circle with its centre at the intersection of two lines. Take the horizontal radius CA as initial line.

Step 2. The perpendicular $B'CB$ gives the angles 90° and 270° .

The diagonals through C give the angles 45° , 135° , 225° , 315° .

By stepping off chords equal to the radius, starting from A , the angles 60° , 120° , 240° , &c., are obtained; and, by starting at B , the angles 30° , 150° , &c., are obtained.

Only the points $P, P', \dots$, on the circumference need be obtained as is shown in the third quadrant; the radii are not needed for drawing the sine graph.

Step 3. Take a point O as origin, some distance along the initial line, and, with a convenient scale, mark off abscissae to represent the angles 30° , 45° , 60° , &c., and, as far as space allows, mark off the negative abscissae.

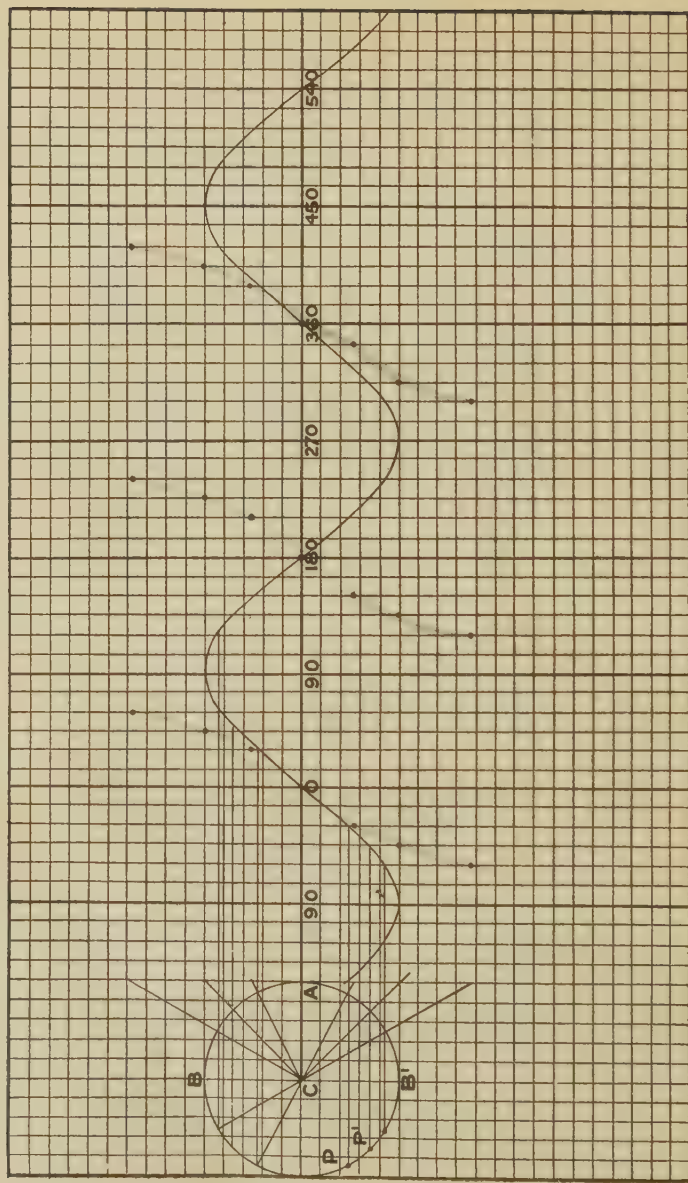
Step 4. Through the points on the circumference draw parallels to the initial line to cut the corresponding ordinates. These points of intersection are points on the graph.

The ordinates of this graph* are proportional to the sines; if we divide by the radius, the actual values of the sines are found.

The sine graph is shown on a larger scale in Fig. X.

19. The tangent graph. To obtain the ordinates for the tangent graph the radii must be produced to meet the tangent to

* The graph of the sine is a wavy or sinuous curve. The name sine is therefore appropriate, although it is improbable that the originators of the name ever drew the graph.



Scale :—for abscissæ, side of small square = 15° ; for ordinates, side = $\cdot 2$.

Fig. IX.

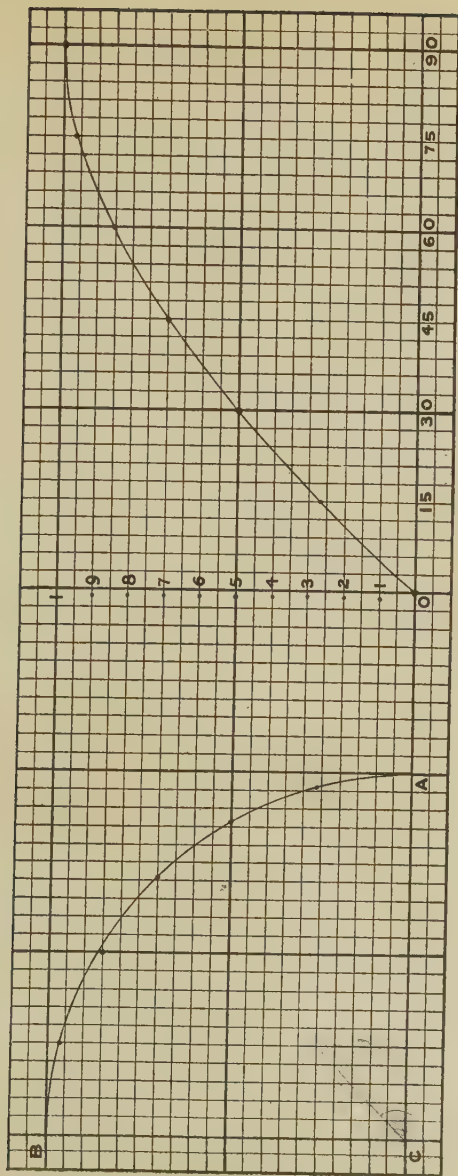


Fig. X.

the circle drawn at A . This is done in Fig. IX, and a few points of the graph are marked, but the graph is not drawn.

The cosine graph. Since the cosine of x is the sine of the complement of x , the student should be able to modify the method for the sine graph so as to obtain the cosine graph.

The secant graph. This is obtained by marking off along the respective ordinates the corresponding values of CT (see Fig. IX). T being the point of intersection of the tangent and produced radius.

Examples II a.

(Answers should be given correct to 2 significant figures.)

By drawing to scale find the trigonometrical ratios of the following angles:

- | | | | |
|------------------|-------------------|-------------------|-------------------|
| 1. 30° . | 2. 49° . | 3. 79° . | 4. 100° . |
| 5. 78° . | 6. 170° . | 7. 250° . | 8. 25° . |
| 9. 300° . | 10. 156° . | 11. -80° . | 12. 415° . |

Find the ratios of the angle AOP when the coordinates of P are

- | | | | |
|-------------|---------------|---------------|----------------|
| 13. (4, 3). | 14. (4, -3). | 15. (-4, -5). | 16. (-4, 3). |
| 17. (3, 2). | 18. (-7, -3). | 19. (-5, 4). | 20. (63, -16). |

The following graphs should be drawn carefully and kept for use :

21-26. A graph on a large scale for each function, for angles from 0° to 90° .

27-32. A graph on a smaller scale for each function for angles from -360° to $+360^\circ$.

33. The blanks in the following table are to be filled with the sign (+ or -) of the respective ratios:

Angle	$0^\circ - 90^\circ$	$90^\circ - 180^\circ$	$180^\circ - 270^\circ$	$270^\circ - 360^\circ$
sine	+	+	-	-
cosine	+	-	-	+
tangent	+	-	+	+

34. If the gradient of a hill, inclined at A° to the horizon, is known, what trigonometrical ratio of the angle is known?

35. Construct an angle whose (i) tangent is 1.45, (ii) sine is .75, (iii) cotangent is 1.45, (iv) secant is 2.7, (v) cosecant is 2.7, (vi) cosine is .75. Measure each angle in degrees.

20. Powers of the Trigonometrical Functions. The square of $\sin A$ is written $\sin^2 A$; and a similar notation is used for other powers and ratios; thus, in general,

$$\sin^n A \text{ means } (\sin A)^n.$$

Inverse notation. There is one exception to the above statement. Suppose $\sin A = a$, then A is an angle whose sine is a . This is written $A = \sin^{-1} a$. Similarly, $\tan^{-1} a$ means an angle whose tangent is a ; and so for the other ratios.

If we wish to express $\frac{1}{\sin A}$ as a power of $\sin A$, we must write $(\sin A)^{-1}$.

Note. Continental mathematicians denote the angle whose sin is x by arc sin x . This notation sometimes occurs in English books.

Example. Determine, by drawing, the angle $\sin^{-1} \frac{2}{3}$.

Step 1. Draw axes OA , OB .

Step 2. Draw circle centre O , radius 3 units.

Step 3. Along OB mark off OK equal 2 units.

Step 4. Through K draw a parallel to OA cutting circle at P' and P . Join OP , OP' .

We now have two angles AOP , AOP' each of which has its sine equal to $\frac{2}{3}$. AOP is 41° , AOP' 139° .

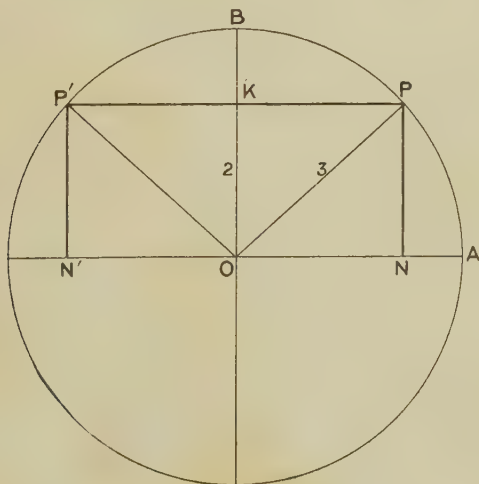


Fig. XI.

It is always understood that $\sin^{-1}a$ means the smallest angle that has the sine equal to a ; and similarly for the other ratios.

Examples II.

1. Find, by drawing to scale, the sine, cosine, and tangent of 30° , 45° , 60° . Verify the results by calculation.

2. The sine of an acute angle is $\frac{2}{5}$; find the cosine, tangent, and secant.

3. The sine of an angle, not acute, is $\frac{2}{5}$; find the cosine and tangent.

4. The cosine of an angle is $\frac{5}{13}$; find the sine and tangent.

5. Draw as many angles as possible having the tangent equal to $\cdot 8$.

6. Given that $\sin 63^\circ = \cdot 89$ find $\cos 63^\circ$ and $\cos 27^\circ$.

7. Find the value of $\sin^2 A + \cos^2 A$, it being known that

$$\sin A = \cdot 3907, \cos A = \cdot 9205.$$

Also find the values of $\tan A$ and $\sec A$.

8. Given $\tan \theta = \frac{1}{5}$, find $\cot \theta$ and $\sec \theta$.

9. Draw and measure an angle A such that (i) $\sin A = -\cdot 5$, (ii) $\cos A = -\cdot 5$, (iii) $\tan A = -\cdot 5$, (iv) $\sec A = -\cdot 5$.

10. Find the value of $\sec^2 \theta - \tan^2 \theta$, when $\sec \theta = 1\cdot 221$ and $\tan \theta = \cdot 7002$. Justify the answer by geometrical reasoning.

11. Are any of the following data inconsistent or impossible? Give reasons for your answers.

(a) $\sin A = \frac{5}{4}$;

(b) $\sec A = \frac{5}{4}$;

(c) $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{7}$;

(d) $\sin A = \cdot 4$, $\cos A = \cdot 6$;

(e) $\sin A = \cdot 6$, $\cos A = \cdot 8$, $\tan A = \cdot 9$;

(f) $\sec A = \cdot 35$, $\tan A = 1\cdot 35$;

(g) $\tan A = 1$; (h) $\sin A = 1$; (i) $\operatorname{cosec} A = 1$.

12. Prove, by means of the definitions in § 12, that

$$\cos A = \sin(90 - A) \text{ and } \tan(\tfrac{1}{2}\pi - \theta) = \cot \theta.$$

13. Find, by drawing to scale, (a) $\sin 36^\circ$ and $\sin 144^\circ$; (b) $\cos 42^\circ$ and $\cos 138^\circ$; (c) $\cos 246^\circ$ and $\cos 66^\circ$.

14. By means of graphs (or otherwise) test the following statements: (a) $\sin(180 - A) = \sin A$; (b) $\cos(180 + A) = -\cos A$; (c) $\sin(90 + A) = \sin A$.

15. By means of graphs find values for $\sin^{-1}\cdot 6$, $\tan^{-1}2\cdot 5$, $\cos^{-1}\cdot 34$, $\cos^{-1}1\cdot 5$.

16. Given $\sin 36^\circ = \cdot 5878$, find $\cos 54^\circ$, $\sin 144^\circ$, $\sin 216^\circ$, $\sin 324^\circ$.
 17. Given $\cos 53^\circ = \cdot 6018$, find $\sin 37^\circ$, $\cos 127^\circ$, $\cos 233^\circ$, $\cos 413^\circ$,
 $\cos 307^\circ$.

18. Prove that $\sin 117^\circ = \cos 27^\circ$.

19. Is it possible to find angles to satisfy the following equations?
 Give reasons.

- | | |
|---|---|
| (i) $\tan \theta = 1$; | (ii) $\cos \theta = \frac{\cdot 934}{\cdot 866}$; |
| (iii) $\sin \theta + \cos \theta = 1$; | (iv) $\sin^2 \theta + \cos^2 \theta = \frac{1}{2}$; |
| (v) $\sec \theta = 3\cdot 1416$; | (vi) $\operatorname{cosec} \theta = \frac{1}{\sqrt{2}}$; |
| (vii) $\sin \theta = 0$; | (viii) $\tan \theta = 100$; |
| (ix) $\cos \theta = 1$; | (x) $\sec \theta = \cdot 78$. |

20. Show that (i) sine and cosine cannot be numerically greater than 1; (ii) tangent and cotangent may be either greater or less than 1; (iii) secant and cosecant cannot be numerically less than 1. Why is the word numerically inserted?

21. Find all the trigonometrical functions of 0° and 90° .

22. (i) Show that the straight line whose equation is $y = mx$ makes an angle $\tan^{-1} m$ with the axis of x .

(ii) What is the tangent of the angle made with the axis of x by the straight line joining the two points (x_1, y_1) and (x_2, y_2) ?

(iii) Show that the equation of the line joining the two points $(x_1, y_1), (x_2, y_2)$ is $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$.

(iv) If the equation of a straight line is $y = mx + c$, give the geometrical meanings of m and c .

23. Show that, if x be any numerical quantity, positive or negative, an angle can be found whose tangent is equal to x .

Show what limitations in value, if any, exist in the case of each of the other trigonometrical ratios.

24. State concisely the changes in the sign and magnitude of $\sin A$ as A increases from 0° to 360° .

25. Define the cosine of an angle of any magnitude, explaining the conventions regarding the signs of the lines referred to in your definitions. Draw the graph of $\cos \theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

26. Define the sine of an angle and find by geometrical reasoning the values of $\sin 45^\circ$, $\sin 90^\circ$, $\sin 135^\circ$.

27. Define the tangent and the versed sine of an angle; and find the greatest and least values which each can have.

28. With ruler and compasses construct an angle whose cosine is $\frac{1}{3}$; also an angle whose cosine is $-\frac{1}{3}$. Calculate the sine of the latter angle to three places of decimals.

29. ABC is a triangle in which AN is the perpendicular from A to BC . If $AB = 2.9$ inches, $AC = 2.5$ inches, $AN = 2$ inches, find the values of $\sin B$, $\cos C$, $\tan B$, $\operatorname{cosec} C$. Calculate the length of BC correct to one decimal place.

30. If A , B , C are the angles of a triangle, express $\sin \frac{1}{2}(A+B)$, $\cos \frac{1}{2}(A+B)$, $\tan \frac{1}{2}(A+B)$ in terms of ratios of $\frac{1}{2}C$.

CHAPTER III

ELEMENTARY FORMULAE

21. Reciprocal Relations.

By definitions, $\sin A = \frac{y}{r}$, $\operatorname{cosec} A = \frac{r}{y}$;

$$\therefore \sin A \operatorname{cosec} A = 1;$$

$$\text{i. e. } \sin A = \frac{1}{\operatorname{cosec} A}, \operatorname{cosec} A = \frac{1}{\sin A}.$$

In a similar way it can be proved that

$$\cos A \sec A = 1, \&c.$$

$$\tan A \cot A = 1, \&c.$$

22. Relations deduced from Pythagoras' Theorem (Prop. 13, p. 10).

In Fig. VII, § 12, we have in all cases

$$ON^2 + NP^2 = OP^2;$$

$$\text{i. e. } x^2 + y^2 = r^2.$$

Three sets of formulae are obtained by dividing in turn by r^2 , x^2 , y^2 .

Divide by r^2 , $\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1;$

but $\frac{x}{r} = \cos A$, $\frac{y}{r} = \sin A$.

Substitute, $\cos^2 A + \sin^2 A = 1.$ *

The equivalent formulae must also be learnt, viz. :

$$\sin^2 A = 1 - \cos^2 A, \sin A = \pm \sqrt{1 - \cos^2 A};$$

$$\cos^2 A = 1 - \sin^2 A, \cos A = \pm \sqrt{1 - \sin^2 A}.$$

In a similar way the student should prove that

$$\tan^2 A + 1 = \sec^2 A,$$

$$\text{and } \cot^2 A + 1 = \operatorname{cosec}^2 A.$$

23. Relation between sine, cosine, and tangent.

$$\begin{aligned}\tan A &= \frac{y}{x} \\ &= \frac{\frac{y}{r}}{\frac{x}{r}}.\end{aligned}$$

Substitute, $\tan A = \frac{\sin A}{\cos A}.$

In a similar way it is proved that

$$\cot A = \frac{\cos A}{\sin A}.$$

24. Identities. By means of the relations proved in the preceding sections, any expression containing trigonometrical functions can be put into a number of forms. It is a useful exercise to prove that two expressions, apparently different, are identical; such exercises serve to fix the relations in the memory and lead to facility in dealing with trigonometrical expressions.

Example. *Prove that $\sec^2 A + \operatorname{cosec}^2 A \equiv \sec^2 A \operatorname{cosec}^2 A$.*

[Express all ratios in terms of sine and cosine.]

Method I.

$$\begin{aligned}\text{L. H. S.} &= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} && \text{by § 21} \\ &= \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A} \\ &= \frac{1}{\sin^2 A \cos^2 A} && \text{using Formula of § 22} \\ &= \frac{1}{\cos^2 A} \times \frac{1}{\sin^2 A} \\ &= \sec^2 A \operatorname{cosec}^2 A. && \text{by § 21}\end{aligned}$$

Q. E. D.

Method II.

$$\begin{aligned}\sec^2 A + \operatorname{cosec}^2 A &= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A} = \frac{1}{\sin^2 A \cos^2 A}, \\ \sec^2 A \operatorname{cosec}^2 A &= \frac{1}{\cos^2 A} \times \frac{1}{\sin^2 A} = \frac{1}{\sin^2 A \cos^2 A}; \\ \therefore \sec^2 A + \operatorname{cosec}^2 A &= \sec^2 A \operatorname{cosec}^2 A. \quad \text{Q.E.D.}\end{aligned}$$

Method III. This method is clumsy, and should be used only if Methods I and II have been tried unsuccessfully.

$$\begin{aligned}\sec^2 A + \operatorname{cosec}^2 A &= \sec^2 A \operatorname{cosec}^2 A, \\ \text{if } \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} &= \frac{1}{\cos^2 A} \cdot \frac{1}{\sin^2 A}, \\ \text{i. e. if } \sin^2 A + \cos^2 A &= 1. \\ \text{But } \sin^2 A + \cos^2 A &\text{ does equal } 1; \\ \therefore \sec^2 A + \operatorname{cosec}^2 A &= \sec^2 A \operatorname{cosec}^2 A.\end{aligned}$$

Note. The introductory ‘if’, or some similar conjunction, is vital to the logical statement of the work and must not be omitted.

25. Elimination. If two equations are satisfied by the same value of a single variable, there must be a relation connecting the constants of the equations; this is also the case when n equations are satisfied by the same values of $n-1$ variables. In order to find this relation we eliminate the variable or variables.

Example. *Eliminate θ from the equations $\sin \theta = a$, $\tan \theta = b$.*

$$\begin{aligned}\text{By formulae} \quad \tan^2 \theta &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \frac{\sin^2 \theta}{1 - \sin^2 \theta}.\end{aligned}$$

$$\text{Substitute} \quad b^2 = \frac{a^2}{1 - a^2},$$

$$\text{i. e.} \quad \frac{1}{a^2} - \frac{1}{b^2} = 1.$$

The result is called the *eliminant* of the original equations.

Examples III a.

1. If $\sin A = \frac{1}{5}$ use formulae to find the remaining ratios. Draw a figure to explain why some of the ratios may be either positive or negative.

2. Given that $\tan \theta = \frac{1}{5}$ find $\cot \theta$ and $\sin \theta$.

3. Find $\sec \theta$ in the following cases:

(i) $\cos \theta = .7921$; (ii) $\tan \theta = 1.352$; (iii) $\operatorname{cosec} \theta = 2.583$.

4. Show how all the ratios may be found when (i) the cosine, (ii) the tangent is known.

5. Prove the following identities:

(i) $\sin A \cot A + \cos A \tan A = \sin A + \cos A$;

(ii) $\tan A + \cot A = \sec A \operatorname{cosec} A$;

(iii) $\sin \theta \tan \theta + \cos \theta \cot \theta = \sec \theta + \operatorname{cosec} \theta - \sin \theta - \cos \theta$;

(iv) $\sec^2 \theta - \operatorname{cosec}^2 \theta = \tan^2 \theta - \cot^2 \theta$;

(v) $1 - 2 \sin^2 A = 2 \cos^2 A - 1$;

(vi) $\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$;

(vii) $(1 - \tan^2 A) \div (1 + \tan^2 A) = \cos^2 A - \sin^2 A$;

(viii) $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$;

(ix) $\sin^3 A - \cos^3 A = (\sin A - \cos A)(1 + \sin A \cos A)$;

(x) $\sin A \cot A \operatorname{cosec} A + \cos A \tan A \sec A = \sec A \operatorname{cosec} A$;

(xi) $\tan X - \tan Y = (\sin X \cos Y - \cos X \sin Y) \div \cos X \cos Y$;

(xii) $(\tan A - \tan B) \div (\cot A - \cot B) = -\tan A \tan B$;

(xiii) $\cos^4 \theta - \sin^4 \theta = 2 \cos^2 \theta - 1$;

(xiv) $(3 - 4 \sin^2 A) \div \cos^2 A = 3 - \tan^2 A$.

6. Prove that $\operatorname{versin} A = 1 - \cos A$, $\operatorname{coversin} A = 1 - \sin A$.

7. Show that the numerical value of $\sin^2 A \div (1 - \cos A)$ diminishes from 2 to 0 as A increases from 0° to 180° , and illustrate your answer by a diagram.

8. Which is greater, the acute angle whose cotangent is $\frac{4}{5}$, or the acute angle whose cosecant is $\frac{5}{4}$?

9. Prove that, if θ is an angle less than 180° for which $1 + \sin \theta = k \cos \theta$, then $\cos \theta = 2k \div (1 + k^2)$; and express $\tan \theta$ in terms of k .

10. Eliminate θ from the following:

(i) $\sin \theta = a$, $\cos \theta = b$;

(ii) $\sin \theta + \cos \theta = a$, $\sin \theta - \cos \theta = b$;

(iii) $\sec \theta - \tan \theta = a$, $\sec \theta + \tan \theta = b$;

(iv) $a \sin \theta + b \cos \theta = p$, $a \sin \theta - b \cos \theta = q$;

(v) $a \sin \theta + b \cos \theta = p$, $a' \sin \theta + b' \cos \theta = p'$.

11. If $a(1 - \sin \theta) = b \cos \theta$, prove that $b(1 + \sin \theta) = a \cos \theta$.

12. If $a(\sec \theta + 1) = b \tan \theta$, prove that $b(\sec \theta - 1) = a \tan \theta$.

13. If $x = a \cos \theta \cos \phi$, $y = a \cos \theta \sin \phi$, $z = a \sin \theta$, eliminate θ and ϕ .

26. Ratios of complementary angles.

Let $XOP = A^\circ$ (Fig. XII) and $XOQ = 90 - A^\circ$; make $OQ = OP$, and let fall QK , PN perpendicular to OX .

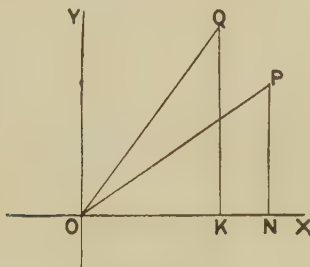


Fig. XII.

Then the triangles QOK , PON are congruent (Prop. 8 c); so that $KQ = ON$ and $OK = NP$.

$$\begin{aligned} \text{Hence } \sin XOQ &= \frac{KQ}{OQ} \\ &= \frac{ON}{OP} \\ &= \cos XOP, \end{aligned}$$

$$\text{i.e. } \sin(90 - A) = \cos A.$$

In a similar way it is proved that*

$$\begin{aligned} \cos(90 - A) &= \sin A \\ \tan(90 - A) &= \cot A \end{aligned}$$

Compare these results with § 14.

What are the corresponding results when angles are measured in radians?

27. Ratios of supplementary angles.

Make $XOP = A^\circ$, and $XOQ = 180 - A^\circ$, so that $QOK = A^\circ$.

* The student is expected to complete these formulae.

Make $OQ = OP$ and let fall the perpendiculars PN, QK .

Then the triangles QOK, PON are congruent (Prop. 8 c) so that $OK = ON$ (in magnitude) and $KQ = NP$. But OK and ON are of opposite sign.

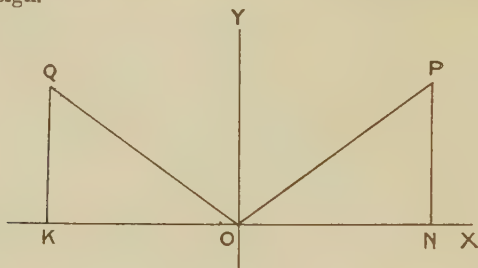


Fig. XIII.

$$\begin{aligned}\text{Hence } \cos XOQ &= \frac{(OK)^*}{OQ} \quad (\text{where } (OK) \text{ denotes the magnitude of } OK \text{ with the proper sign prefixed}) \\ &= \frac{-ON}{OP} \\ &= -\cos XOP,\end{aligned}$$

$$\text{i. e. } \cos (180 - A) = -\cos A.$$

In a similar way it is proved that

$$\begin{aligned}\sin (180 - A) &= \sin A \\ \tan (180 - A) &= -\tan A\end{aligned}$$

The student should go through the proof in cases when A is not acute.

28. Ratios of negative angles.

Make $XOP = +A^\circ$ and $XOQ = -A^\circ$.

Then $XOP = XOQ$ in magnitude.

Make $OQ = OP$.

Join PQ cutting OX at N .

Then in the triangles PON, QON

$$\begin{cases} OP = OQ, \\ ON \text{ is common,} \\ \text{included angle } NOP = \text{included angle } NOQ. \end{cases}$$

$\therefore NP = NQ$ in magnitude,

and $ONP = ONQ$, so that PQ is perpendicular to OX .

* In writing it is usual to use the symbol $\overline{OK}$ to denote length preceded by correct sign; it is more convenient to print (OK) .

$$\begin{aligned}
 \text{Hence } \cos XOQ &= \frac{(ON)}{OQ} \\
 &= \frac{(ON)}{OP} \\
 &= \cos XOP,
 \end{aligned}$$

$$\text{i.e. } \cos(-A) = \cos A.$$

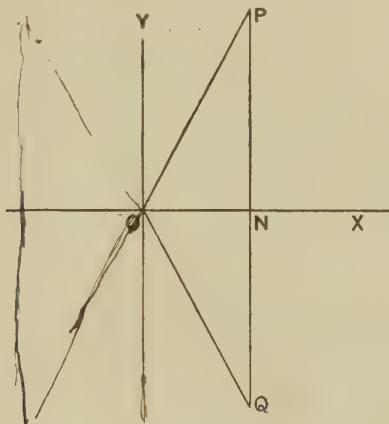


Fig. XIV.

In a similar way it is proved that

$$\sin(-A) = -\sin A$$

$$\tan(-A) = -\tan A$$

The student should also work out the ratios of $90 + A$, $180 + A$, $270 - A$, &c., (i) when A is acute, (ii) when A is not acute.

29. By means of the last three sections the ratios of any angle can be expressed in terms of the ratios of an acute angle not greater than 45° . For example

$$\cos 139^\circ = \cos(180^\circ - 41^\circ) = -\cos 41^\circ,$$

$$\cos 246^\circ = \cos(-114^\circ)$$

$$= \cos(114^\circ)$$

$$= \cos(180^\circ - 66^\circ)$$

$$= -\cos 66^\circ$$

$$= -\cos(90^\circ - 24^\circ)$$

$$= -\sin 24^\circ.$$

It is usually easy to work directly from the figure; thus in Fig. XV, where $XOP = 246^\circ$ and $XOQ = 66^\circ$,

$$\begin{aligned}\cos 246^\circ &= \frac{(ON)}{OP} \\ &= -\frac{OK}{OQ} \\ &= -\cos 66^\circ \\ &= -\sin 24^\circ.\end{aligned}$$

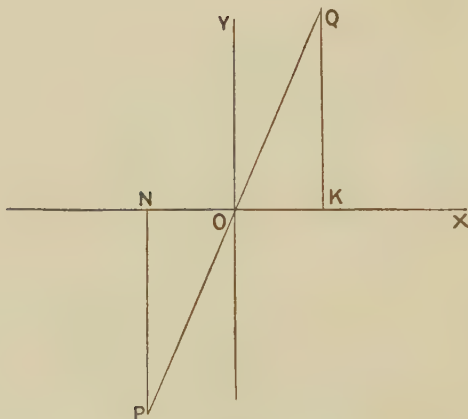


Fig. XV.

30. Ratios of 0° and 90° .

If $XOP = 0^\circ$, P and N coincide; so that $NP = 0$, $ON = OP$.

Hence

$$\begin{aligned}\sin 0^\circ &= \frac{NP}{OP} = 0, \\ \cos 0^\circ &= \frac{ON}{OP} = 1, \\ \tan 0^\circ &= \frac{NP}{ON} = \frac{0}{ON} = 0.\end{aligned}$$

If $XOP = 90^\circ$, then PN falls along the y axis and N coincides with the origin O . In this case $NP = OP$ and $ON = 0$.

Hence $\sin 90^\circ = \frac{NP}{OP} = 1,$
 $\cos 90^\circ = \frac{ON}{OP} = 0,$
 $\tan 90^\circ = \frac{NP}{ON} = \frac{NP}{0} = \infty.*$

31. Ratios of 30° , 45° , 60° .

Make XOP equal to 30° , Fig. XVI.

Let fall PN perpendicular to OX .

Make XOQ equal to 30° in magnitude, and produce PN to meet OQ in Q .

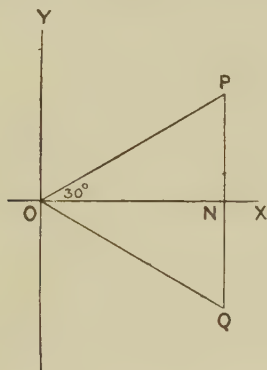


Fig. XVI.

Then, by Prop. 8c, the triangles PON , QON are congruent. It follows that the triangle OPQ is equilateral.

* The symbol ∞ means 'infinity'. Infinity is not a definite number, but is a term applied to any number greater than we can imagine.

Consider the value of $1/x$ as x gets smaller and smaller.

$$\frac{1}{.1} = 10, \quad \frac{1}{.001} = 1000, \quad \frac{1}{.000001} = 1000000.$$

As x diminishes, $1/x$ increases, and, by making x sufficiently small, we can make $1/x$ exceed any assigned value however great. This is expressed thus: when $x = 0$, $1/x = \infty$. Or more generally, if a is a constant, then $a/x = \infty$ when $x = 0$.

Hence $PN = \frac{1}{2}PQ$ since $PN = QN$
 $= \frac{1}{2}OP$ since $PQ = OP$;

also $ON^2 = OP^2 - PN^2 = OP^2 - \frac{1}{4}OP^2 = \frac{3}{4}OP^2$.

$\therefore ON = \frac{\sqrt{3}}{2} OP$.

Hence $\sin 30^\circ = \frac{NP}{OP} = \frac{1}{2} = \cdot 5,$

$\cos 30^\circ = \frac{ON}{OP} = \frac{\sqrt{3}}{2} = \cdot 866,$

$\tan 30^\circ = \frac{NP}{ON} = \frac{1}{\sqrt{3}} = \cdot 577.$

Similarly $\sin 60^\circ = \frac{\sqrt{3}}{2} = \cdot 866,$

$\cos 60^\circ = \frac{1}{2} = \cdot 5,$

$\tan 60^\circ = \sqrt{3} = 1\cdot 732.$

As an exercise the student should find the values of the ratios of 45° .

$\sin 45^\circ =$

$\cos 45^\circ =$

$\tan 45^\circ =$

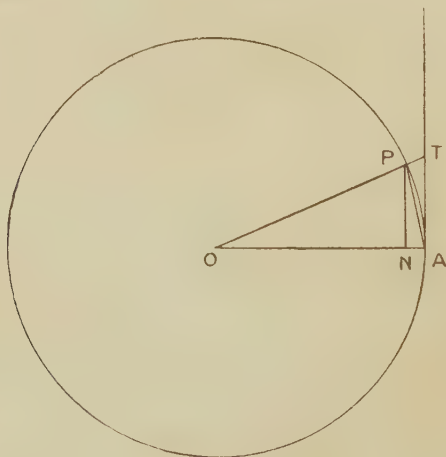


Fig. XVII.

32. The very small angle.

In Fig. XVII let the circular measure of the angle AOP be θ .

Then arc $AP = r\theta$, $NP = r \sin \theta$, $AT = r \tan \theta$.

Hence Area of triangle $AOP = \frac{1}{2} OA \cdot NP$ (Prop. 16)

$$= \frac{1}{2} r^2 \sin \theta;$$

Area of sector $AOP = \frac{1}{2} r^2 \theta$; (§ 7)

Area of triangle $AOT = \frac{1}{2} OA \cdot AT$

$$= \frac{1}{2} r^2 \tan \theta. \quad (\text{Prop. 16})$$

But, if AOP is an acute angle,

Triangle $AOP < \text{sector } AOP < \text{Triangle } AOT$,

$$\text{i.e. } \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan \theta,$$

$$\text{i.e. } \sin \theta < \theta < \tan \theta.$$

This relation is true for any acute angle; if we multiply throughout by r we have

$$r \sin \theta < r\theta < r \tan \theta,$$

$$\text{i.e. } NP < \text{arc } AP < AT.$$

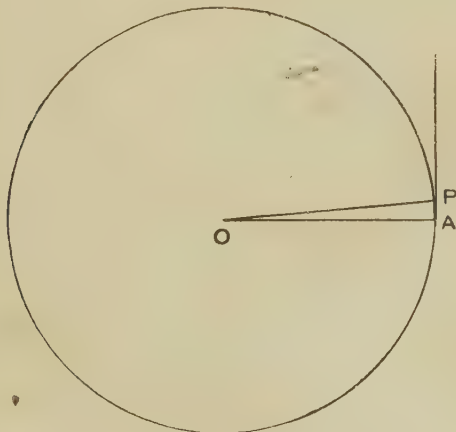


Fig. XVIII.

But, as the angle diminishes, these three lengths more and more nearly coincide; and are practically indistinguishable when the angle is very small. This is shown in Fig. XVIII, which also shows that ON is indistinguishable from OA .

Hence, when θ is very small, there is very slight error in saying $NP = \text{arc } AP = AT$, and $ON = OA$.

Substituting the trigonometrical values for the lengths of these lines, we have

$$\sin \theta = \theta = \tan \theta \quad \text{and} \quad \cos \theta = 1,$$

when θ , the circular measure of the angle, is small.

This may also be expressed thus: The limit of $\frac{\sin \theta}{\theta}$ or of $\frac{\tan \theta}{\theta}$, when θ is zero, is 1; or in symbols

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1.*$$

33. Error involved. Whatever be the value of θ , it has been shown that

$$\cos^2 \theta + \sin^2 \theta = 1.$$

Using the above approximations, we have

$$1 + \theta^2 = 1.$$

This last statement is true only when θ^2 is so small as to be negligible. Hence

If θ is so small that θ^2 may be neglected, we may say that

$$\sin \theta = \theta, \quad \cos \theta = 1, \quad \tan \theta = \theta.$$

It is shown in Higher Trigonometry that $\sin \theta = \theta$ gives correct results if $\frac{1}{2}\theta^3$ is negligible.

Example. *If accuracy is required to four decimal places, find the sine of 1 degree.*

$$1^\circ = \frac{1}{180} \pi \text{ radians} = .01745 \text{ radian}$$

$$.01745^2 = .000295. \quad (\text{We are not, therefore, justified in saying } \cos 1^\circ = 1.)$$

$$.01745^3 = .000005. \quad (\text{This does not affect the first four places so we may use the approximation } \sin \theta = \theta.)$$

Hence $\sin 1^\circ = .0175$, correct to four decimal places.

Examples III b.

1. Write down the sine, cosine, and tangent of

(i) 150° , 240° , 330° , 840° ;

(ii) 60° , 300° , 135° , 225° ;

(iii) 180° , 270° , 405° , 210° .

2. Find the secant and cosecant of 60° , 45° , 120° , 225° .

3. Use the definitions of § 14 to find the ratios of $180 - A$.

* For explanation of the word 'limit' see *School Algebra*, Part ii, p. 446.

4. Correct, if necessary, the following statements :

$$\sin(180 - A) = \cos A; \cos(270 + A) = -\cos A;$$

$$\tan(180 + A) = \tan A; \sec(90 - A) = \sec A;$$

$$\cot(90 + A) = \cot A.$$

5. In a right-angled triangle the hypotenuse is 5 feet long and one of the angles is 60° ; find the lengths of the other two sides.

6. A ladder 25 feet long is leaning against a wall and is inclined 45° to the horizontal; how far up the wall does it reach?

7. Find $\sin 1'$, correct to 3 significant figures.

8. Find $\sin 10'$, $\cos 10'$, $\tan 10'$ correct to 5 decimal places.

9. What angle does a halfpenny (diameter 1 inch) subtend at the eye when at a distance of 10 feet?

10. A post 25 feet high subtends an angle of $30'$ at a certain point on the ground. How far from the post is the point?

11. Find approximately the distance of a tower which is 51 feet high and subtends at the eye an angle $5\frac{3}{11}'$.

12. Prove that

$$\tan^2 60^\circ - 2 \tan^2 45^\circ = \cot^2 30^\circ - 2 \sin^2 30^\circ - \frac{3}{4} \operatorname{cosec}^2 45^\circ.$$

13. Find approximately the number of minutes denoting the inclination to the horizon of an incline which rises $5\frac{1}{2}$ feet in 420 yards.

14. In any triangle show that

$$\cos(A + B) = -\cos C, \sin(B + C) = \sin A, \tan(B + C) = -\tan A.$$

Write down the other similar relations.

Oral Examples.

Fill in the right-hand sides of the following equalities:

1. (i) $\sin^2 \theta =$

(ii) $\sin 45^\circ =$

(iii) $\cos 135^\circ =$

(iv) $\tan \frac{1}{4} \pi =$

(v) $\sin A \cot A = \cos A$

3. (i) $\cos^2 60^\circ + \sin^2 60^\circ =$

(ii) $\operatorname{cosec}^2 C =$

(iii) $\cot \frac{1}{2} \pi =$

(iv) $\cos(180 - A) =$

(v) $\cot^2 \theta =$

2. (i) $\sec^2 A - \tan^2 A = 1$

(ii) $\cos 60^\circ =$

(iii) $\tan \theta =$

(iv) $\sin(180 - A) =$

(v) $\sec(90 - B) =$

4. (i) $\cos \theta \tan \theta =$

(ii) $1 - \sin^2 x =$

(iii) $\tan 210^\circ =$

(iv) $\cos^{-1} \frac{1}{2} =$

(v) $\cos^2 \frac{1}{4} \pi + \sin^2 \frac{1}{4} \pi =$

5. (i) $1 + \cot^2 A =$
 (ii) $\sin \theta \cot \theta =$
 (iii) $\sin (180 + A) =$
 (iv) $\cos^2 63^\circ + \sin^2 63^\circ =$
 (v) $\tan 330^\circ =$
7. (i) $\tan \frac{1}{4}\pi =$
 (ii) $\sec 60^\circ =$
 (iii) $\sin^2 \frac{1}{2}A + \cos^2 \frac{1}{2}A =$
 (iv) $\tan (\frac{3}{2}\pi + \theta) =$
 (v) $\tan 135^\circ =$
9. (i) $\tan 150^\circ =$
 (ii) $\cos \theta / \sin \theta =$
 (iii) $\cos 90^\circ =$
 (iv) $\sec 240^\circ =$
 (v) $\cot (180 - A) =$
11. (i) $\tan 1200^\circ =$
 (ii) $\tan (180^\circ + A) =$
 (iii) $\tan \frac{1}{2}\pi =$
 (iv) $\tan 15^\circ \cot 15^\circ =$
 (v) $\tan^{-1}(-1) =$
6. (i) $\cos^{-1} \frac{\sqrt{2}}{2} =$
 (ii) $\cos 225^\circ =$
 (iii) $\cos (90 + A) =$
 (iv) $\cos (-\theta) =$
 (v) $\cos (180 - B) =$
8. (i) $\sin (360 - A) =$
 (ii) $\sin^{-1} 2 =$
 (iii) $\sec^2 \frac{1}{5}\pi - \tan^2 \frac{1}{5}\pi =$
 (iv) $\cos 0^\circ =$
 (v) $\operatorname{cosec} 120^\circ =$
10. (i) $\sec 150^\circ =$
 (ii) $\cos (360^\circ - A) =$
 (iii) $\cos^{-1} \sqrt{3} =$
 (iv) $\sin 77^\circ \cot 77^\circ =$
 (v) $\tan \frac{7}{4}\pi =$
12. (i) $\cos^2 23^\circ + \cos^2 67^\circ =$
 (ii) $\cos (270^\circ + B) =$
 (iii) $\sin^{-1} .4 + \cos^{-1} .4 =$
 (iv) $\sin (-\phi) =$
 (v) $\sin 225^\circ =$

Examples III.

1. Prove, from first principles, that $\sin (90 + A) = \cos A$, $\cos (180 + A) = -\cos A$, $\tan (360 - A) = -\tan A$.

2. Show that $\sin (180 - A) = \sin A$, when A is (i) obtuse, (ii) between 180° and 270° , (iii) between 270° and 360° .

3. Show that $\cos (90 - A) = \sin A$, when A is (i) obtuse, (ii) between 180° and 270° , (iii) between 270° and 360° .

4. Show that $\tan (180 + A) = \tan A$, when A is (i) obtuse, (ii) between 180° and 270° , (iii) between 270° , and 360° .

5. Give 6 different solutions of each of the following equations:

$$(i) \sin A = \frac{1}{2}; \quad (ii) \sin A = \frac{\sqrt{2}}{2}; \quad (iii) \cos \theta = \frac{\sqrt{3}}{2};$$

$$(iv) \tan A = 1; \quad (v) \cos A = -\frac{1}{2}; \quad (vi) \sin \theta = -\frac{\sqrt{3}}{2}.$$

6. Show that all angles having the same sine as A are included in one or other of the forms: $180n + A$, if n is an even integer,

$180n - A$, if n is an odd integer; and that these are included in the single form $180n + (-1)^n A$ where n is any integer, positive or negative.

7. Show that all angles having the same cosine as A are included in the form $360n \pm A$, where n is any integer.

8. Show that all angles having the same tangent as A are included in the form $180n \pm A$, where n is any integer.

9. What do the forms of the three previous examples become when the angle is measured in radians?

10. If a small angle equals A° , what is the value of $\sin A$?

11. Show that using the approximation $\sin \theta = \theta$ is equivalent to regarding a circle as a polygon with a large number of sides.

12. What do the following equalities become when the angle θ is so small that θ^2 is negligible?

(i) $\sin 2\theta = 2 \sin \theta \cos \theta$;

(ii) $\cos 2\theta = 1 - 2 \sin^2 \theta$;

(iii) $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$;

(iv) $C = G \tan \theta$;

(v) $v^2 = 4gr \sin^2 \frac{1}{2}\theta$.

13. Two strings are tied to two pegs A and B in the same horizontal line, and knotted together at C ; when the strings are pulled tight, it is found that AC is 18 inches long and that the angles CAB, CBA are 30° and 60° respectively; how far apart are the pegs and how far is C from AB ?

14. An inclined plane, length 4 feet, is inclined at 30° to the horizontal, what is the length of the base?

15. A pendulum is held so as to make an angle of 30° with the vertical, what is then the distance of the end of the pendulum from the vertical line through point of support?

16. Prove the following identities:

(i) $\cot^2 A \cos^2 A = \cot^2 A - \cos^2 A$;

(ii) $\sec^2 A - \sin^2 A = \tan^2 A + \cos^2 A$;

(iii) $\sin \theta (\operatorname{cosec} \theta - \sin \theta) = \cos^2 \theta$;

(iv) $(\cos A + \operatorname{cosec} A)(\sin A + \sec A)$
 $= 2 + \sin A \cos A + \sec A \operatorname{cosec} A$;

(v) $(\cos A + \sec A)(\sin A + \operatorname{cosec} A)$
 $= \sin A \cos A + 2 \sec A \operatorname{cosec} A$;

(vi) $\sec A - \sin A \tan A = \cos A$;

(vii) $(\sec A - \operatorname{cosec} A)(\sin A + \cos A) + \sec^2 A \cot A = 2 \tan A$.

17. (i) If θ and ϕ differ by $\frac{1}{2}\pi$, prove that $\tan \theta \tan \phi = -1$;

(ii) Show that the lines whose equations are, respectively, $y = mx$ and $y = m'x$, are at right angles if $mm' = -1$;

(iii) Show that the graphs of the equations $ax + by + c = 0$, $a'x + b'y + c' = 0$ are at right angles if $aa' + bb' = 0$, and are parallel if $a/a' = b/b'$.

18. If $\tan \theta = b/a$, find the value of $a \cos \theta + b \sin \theta$.

19. If $\tan^3 \theta = b/a$, show that $a/\cos \theta + b/\sin \theta = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

20. Give a general formula for all values of A which satisfy the equation $\cos A = -1$.

21. If $a \sin^2 \theta + b \cos^2 \theta = c$ and $a \cos^2 \theta + b \sin^2 \theta = d$, prove that $a + b = c + d$.

22. From the vertex C of an equilateral triangle ABC a perpendicular CD is let fall on AB ; DC is produced to E so that CE equals CA , and AE is drawn. From the resulting figure find the sine, cosine, and tangent of 15° and 75° .

23. A is an angle between 180° and 270° , also $\cos A = -\frac{3}{5}$; find the value of $\operatorname{cosec} A + \tan A$.

24. Define the cosine of an angle of any magnitude and express the cosine of an angle between 180° and 270° in terms of each of the other trigonometrical ratios.

If $\cos \theta = -\frac{5}{13}$, find $\sin \theta$, $\sec \theta$, $\cot \theta$, and explain any double signs which occur in your answer.

25. Prove the following identities:

(i) $(\sin A \cos B + \cos A \sin B)^2 + (\cos A \cos B - \sin A \sin B)^2 = 1$;

(ii) $\sin^6 \theta + \cos^6 \theta = 1 - 3 \sin^2 \theta \cos^2 \theta$;

(iii) $\cot A - \tan A = \sec A \operatorname{cosec} A (1 - 2 \sin^2 A)$;

(iv) $(1 - \sin A - \cos A)^2 = 2 (1 - \sin A) (1 - \cos A)$;

(v) $(2 \cos A - \sec A) \div (\cos A - \sin A) = 1 + \tan A$.

(vi) $(3 \sin \theta \cos^2 \theta - \sin^3 \theta)^2 + (\cos^3 \theta - 3 \cos \theta \sin^2 \theta)^2 = 1$;

(vii) $\sec^6 \theta - \tan^6 \theta = 1 + 3 \tan^2 \theta \sec^2 \theta$;

(viii) $\operatorname{versin} (270^\circ + A) \cdot \operatorname{versin} (270^\circ - A) = \cos^2 A$.

26. Prove that

$$\cos (180^\circ - A) = -\cos A, \text{ and } \cos (90^\circ + A) = -\sin A.$$

For what values of A is $\tan A = \sqrt{3}$ and $\sec A = -2$?

27. Solve for x the equations:

(i) $x^2 + 2x \sec \alpha + 1 = 0$;

(ii) $x^2 + 2x \cos \alpha = \sin^2 \alpha$;

(iii) $x^2 + (\tan \alpha + \cot \alpha)x + 1 = 0$.

28. Prove that the number of seconds in an angle whose circular measure is unity is 206,265.

The moon subtends at the eye of an observer an angle of $30'$, its distance is 240,000 miles, find its radius.

29. If $\tan^2 \theta = \frac{5}{4}$, find $\text{versin } \theta$, and explain the double result.

30. Eliminate θ from

$$(i) a \tan \theta + b \cot \theta = c, a' \tan \theta + b' \cot \theta = c';$$

$$(ii) a \tan \theta + b \sin \theta = c, a' \tan \theta + b' \sin \theta = c'.$$

Revision Examples A.

1. Define the tangent of an angle. From your definition find $\tan 45^\circ$ and $\tan 135^\circ$, and prove that $\tan(\frac{1}{2}\pi - \theta) = \cot \theta$.

2. A surveyor goes 10 chains in a direction 35° S. of E., then 7.8 chains 14° E. of S.; then 5.6 chains 10° N. of W. Find by drawing how far he is now from his starting point.

3. Prove the relation $1 - \sin^2 A = \cos^2 A$ for the case where A lies between 90° and 180° .

Show that $(\sin A + \cos A)^4 = 1 + 4 \sin A \cos A + 4 \sin^2 A - 4 \sin^4 A$.

4. The gradient of a railway is 1 in 270; find the inclination to the horizontal to the nearest second.

5. When the sun's altitude is 60° , find the length of the shadow cast by a vertical rod whose length is 10 feet.

6. Draw the graph of $\cos x$ between $x = 15^\circ$ and $x = 135^\circ$ without using tables.

7. Explain how to find the length of the arc of a circle of given radius, when the angle subtended at the centre is given in degrees.

A wheel, radius $4\frac{1}{2}$ feet, rolls along the ground; what horizontal distance does the centre travel when the wheel turns through 157° ?

8. Why is the secant so called? Prove that the secant is the reciprocal of the cosine.

Given $\sec A = 2\frac{1}{2}$, find $\tan A$ and $\sin A$.

9. Show that the graph of the straight line $y = 2x - 5$ is inclined to the axis of x at an angle $\tan^{-1} 2$. Verify this by a careful drawing.

10. Trace the changes in $\sin \theta$ as θ changes from 0° to 360° and exhibit these changes by means of a graph.

11. Find the smallest angle which satisfies the equation

$$3 \cos \theta + 2 \sin^2 \theta = 0.$$

Give also four other solutions.

12. If $\sin A = \frac{2}{5}$, prove that $\sec A + 1/\cot A = 2$.

13. What is a radian? Prove the formula

$$\text{arc} = r \times \theta.$$

Show that if θ is small, $\sin \theta = \theta$ approximately.

14. Show that $\tan(180n + A) = \tan A$ where n is any integer.

If $\tan 3A = \sqrt{3}$, state three possible values for A that do not differ by 360° .

15. Find the value of the expression $\operatorname{cosec} A - \frac{5}{8} \cot A$, if $\sin A = \frac{6}{5}$ (i) when A is acute, (ii) when A is obtuse.

16. Prove the identity $2 \sin A \cos A = (2 \tan A) \div (1 + \tan^2 A)$.

17. In a triangle ABC , $C = 90^\circ$, $AB = 15$, $\sin A = .37$; find the length of AC and BC .

18. Criticize the following statements:

(a) $\sin^2 \theta = 4$;

(b) $\sin \theta \tan \theta = 1$;

(c) $\sin^{-1}(-.3) = 170^\circ$; (d) $\sin \theta + \cos^2 \theta / \sin \theta = \tan \theta$.

19. Explain clearly what is meant by latitude.

A place has latitude 30°N. , what is its distance from (i) the earth's axis, (ii) the Equator, measured along the surface? (Radius of earth = 4000 miles.)

20. Give a definition of cosine that applies to angles of any size. Prove that $\cos(180 - A) = -\cos A$.

If $\sin A = \frac{1}{2}$ and A is obtuse, find $\cos A$.

21. Prove that

$$(\cos A \cos B + \sin A \sin B)^2 + (\sin A \cos B - \cos A \sin B)^2 = 1.$$

22. Draw the graph of $y = \sec x$ from $x = 0^\circ$ to $x = 180^\circ$.

23. What is meant by the statement that $\tan 90^\circ = \infty$?

Is $\sec 90^\circ$ equal to $\tan 90^\circ$? Give reasons.

24. Construct an angle A such that $\cos A = -\frac{2}{3}$ and $\tan A$ is positive.

25. Name the points of the compass between West and South.

How many degrees are there in the angle between SW. by S. and S. by E.?

26. Find the values of $\sin 45^\circ$, $\cos^{-1} \frac{1}{2}$, $\tan \frac{1}{4}\pi$.

27. Prove by means of a figure that

$$\sin^2 A + \tan^2 A = \sec^2 A - \cos^2 A.$$

Is this true when the angle is measured in radians? Give reasons.

28. Construct an angle such that its tangent $= \frac{5}{4}$ and its versine is greater than unity.

29. Prove the identity $\cos^2 A - \sin^2 A = (\cot^2 A - 1) \div \operatorname{cosec}^2 A$.

30. Find the value of $\tan \theta$ from the equation

$$3 \tan^2 \theta = 2\sqrt{3} \tan \theta - 1.$$

Hence find three different values of θ that satisfy the equation.

31. Write down six positive angles which have the same cosine as the angle α ; and find the positive values of θ less than two right angles which satisfy the equation

$$\sin 4\theta = \cos 5\theta.$$

32. Show how to find by calculation the value of $\sin 30^\circ$ correct to four decimal places.

Verify, by substitution,

$$(i) \sin 60^\circ = 2 \sin 30^\circ \cos 30^\circ;$$

$$(ii) \sin 120^\circ - \sin 60^\circ = 2 \cos 90^\circ \sin 30^\circ;$$

$$(iii) \cos 60^\circ - \cos 120^\circ = 2 \sin 30^\circ \sin 90^\circ.$$

33. Prove the identities:

$$(i) \operatorname{cosec}^2 A - \cotan^2 A = 1;$$

$$(ii) \frac{5 + 13 \sin \theta}{12 + 13 \cos \theta} + \frac{12 - 13 \cos \theta}{5 - 13 \sin \theta} = 0.$$

34. A steamer travels along the equator from longitude $37\frac{1}{2}^\circ$ W. to longitude $5^\circ 30'$ E. in 4 days. What is the distance travelled in nautical miles? What was her average rate in knots?

35. What is meant by the chord of an angle?

For which angle is the chord equal to unity?

Explain how to draw an angle when a table of chords is given.

36. Express the following ratios as ratios of angles not greater than 45° :

$$\sin 172^\circ, \cos 412^\circ, \tan 246^\circ, \sec 76^\circ, \operatorname{cosec} 147^\circ, \sec 236^\circ,$$

$$\cot 138^\circ, \cosine 150^\circ, \sin 67^\circ, \tan 102^\circ.$$

37. If the circumferences of the quadrants of two circles be divided similarly to the right angles they subtend, what would be the radius of a circle divided according to the French scale, in which the length of the arc of one grade would be equal to the length of the arc of one degree on a circle whose radius was 18 feet?

38. Point out which of the trigonometrical functions are never numerically less than unity, and which may be either less or greater than unity.

Express the numerical values of $\sin 135^\circ$ and $\tan 150^\circ$ with their proper signs.

39. If n be a positive whole number, show that the angles

$$(2n \cdot 180^\circ + A) \text{ and } \{(2n + 1)180^\circ - A\}$$

have the same sine as A .

Express these in a single formula.

40. Distinguish carefully between $(\sin A)^{-1}$ and $\sin^{-1} A$.

Show that $\cos^{-1} \frac{1}{2} + 2 \sin^{-1} \frac{1}{2} = 120^\circ$.

41. Trace the changes in sign and magnitude of the expression $\cos x - \sin x$ as x increases from 0 to 2π . Illustrate your answer by a graph.

42. A church spire, whose height is known to be 45 feet, subtends an angle of $9'$ at the eye; find its distance approximately.

(43.) What is meant by $\tan^{-1} m$?

If $y = mx + c$ represents a straight line, state the geometrical interpretation of the coefficients m and c ?

What is the angle between the lines whose equations are

$$y = x - 4, \quad y = \sqrt{3}x + 2?$$

44. Show that the equation of the line joining the points

$$(x_1, y_1), (x_2, y_2) \text{ is } (y - y_1) \div (y_1 - y_2) = (x - x_1) \div (x_1 - x_2).$$

45. Find the equation of a line passing through the origin and (i) parallel to, (ii) perpendicular to, the line whose equation is $y = mx + c$.

Deduce the conditions that the two lines whose equations are $ax + by + c = 0$, $a'x + b'y + c' = 0$, should be (i) parallel, (ii) perpendicular.

46. Find the equation of the line joining the origin to the point P whose co-ordinates are (x', y') .

Find the equation of the line perpendicular to OP and passing through P .

Hence show that the equation of the tangent to a circle at the point x', y' is $xx' + yy' = r^2$, the equation of the circle being $x^2 + y^2 = r^2$.

47. If (r, θ) are the polar co-ordinates of a point, what locus is represented by

(i) $r = 3$, (ii) $\theta = \frac{1}{3}\pi$, (iii) $r \cos \theta = 5$, (iv) $r = 5 \cos \theta$?

48. If (x, y) are the Cartesian co-ordinates, (r, θ) the polar co-ordinates, of the same point, what relations connect them?

Express the equations of the previous example in Cartesian co-ordinates.

Express (i) $x^2 + y^2 - 4x + 5y = 7$, (ii) $3x + 4y = 5$ in polar co-ordinates.

49. The sum of two angles is 3 radians, their difference is 10 degrees. Find each angle in degrees, assuming that $43\pi = 135$.

50. A ring, 10 inches in diameter, is suspended from a point one foot above its centre by six equal strings attached to its circumference at equal intervals. Find the angle between two consecutive strings.

CHAPTER IV

USE OF TABLES

34. It has been shown in the previous chapters that the trigonometrical ratios of any angle may be found roughly by drawing to scale or by means of graphs. By methods which are explained in more advanced books on Trigonometry, the ratios can be calculated to any required degree of accuracy. There are many collections of tables published, containing not only the actual trigonometrical ratios (the natural functions as they are called) but also the logarithms of these ratios. These collections differ slightly in their arrangement, but the following general remarks apply to most of them.

35. Since any ratio of any angle is equal in magnitude to the same ratio of some angle less than 90° , it is necessary to tabulate the ratios only for angles between 0° and 90° . Thus

$$\begin{aligned}\sin 156^\circ &= \sin(180^\circ - 24^\circ) = \sin 24^\circ, \\ \cos 215^\circ &= \cos(180^\circ + 35^\circ) = -\cos 35^\circ.\end{aligned}$$

But the tables may be made even shorter, for any function of an angle between 45° and 90° is equal to the complementary function of an angle less than 45° . Thus

$$\begin{aligned}\sin 76^\circ &= \sin(90^\circ - 14^\circ) = \cos 14^\circ, \\ \tan 69^\circ &= \tan(90^\circ - 21^\circ) = \cot 21^\circ.\end{aligned}$$

This fact is used in two different ways. Some tables give all the ratios for angles from 0° to 45° ; so that if, for instance, $\sin 72^\circ$ is required, it must be looked up as $\cos 18^\circ$. Other tables give the values of sine, tangent, and secant for angles from 0° to 90° ; in this case, cosine, cotangent, and cosecant must be looked for as the sine, tangent, and secant respectively of the complementary angle.

The slight mental work involved is avoided by giving each column a "footing" as well as a heading. Thus '26892 is, in some tables, found on a page headed Natural Sines, on a level with 15° in the extreme left-hand column and under $36'$, i.e. $'26892 = \sin 15^\circ 36'$.

But the same page has Natural Cosines at the bottom, '26892 is on same level as 74° in the extreme right-hand column and above 24', i.e. $\cdot 26892 = \cos 74^\circ 24'$.

A few minutes' inspection will make the arrangement of any set of tables quite clear.

36. Logarithmic Functions. Since the sine and cosine cannot be greater than unity, their logarithms cannot be greater than zero; hence these logarithms have a negative characteristic. In order to avoid difficulties of printing it has been the custom to add 10 to all these logarithms, and to the other logarithmic functions. The values thus tabulated are called **Tabular Logarithms** and are denoted in writing by L, thus $L \tan 75^\circ = \log \tan 75^\circ + 10$.

Some of the modern tables give the ordinary logarithms with the negative characteristics.

When tabular logarithms are used it is advisable to subtract 10 mentally and to work with the correct logarithm.

37. Interpolation. It is impossible to give the ratios for all angles. Four-figure tables usually give values for every $6'$, seven-figure tables for every $1'$. Intermediate values may, in some tables, be found from side columns giving the differences, as in the case of ordinary logarithms. If these side columns are not given, the method of **proportional parts** * must be used. This method is equivalent to assuming that the graph of the tabulated function may be treated as a straight line for portions lying between the points corresponding to two consecutive tabulated values. The practical use is easily followed from an example or two.

Example i. *Given that*

$$\sin 28^\circ 9' = \cdot 4717815, \text{ and } \sin 28^\circ 10' = \cdot 4720380,$$

find $\sin 28^\circ 9' 43''$.

$$\sin 28^\circ 10' = \cdot 4720380,$$

$$\cdot 000004275 \times 43$$

$$\sin 28^\circ 9' = \cdot 4717815.$$

$$\cdot 00017100$$

$$\text{Increase for } 60'' = \cdot 0002565;$$

$$1282$$

$$\therefore \text{Increase for } 43'' = \frac{43}{60} \times \cdot 0002565$$

$$\cdot 00018382$$

$$= \cdot 0001838;$$

$$\therefore \sin 28^\circ 9' 43'' = \cdot 4719653.$$

* For a fuller treatment see *School Algebra*, Part II, p. 376.

In practice the zeros are omitted as in the following example.

Example ii. Given that $\log \cos 73^\circ 15' = \bar{1} \cdot 4058617$, and $\log \cos 73^\circ 16' = \bar{1} \cdot 4053816$, find the angle when the log cosine is $\bar{1} \cdot 4056348$.

Denote the angle by $73^\circ 15' x''$.

$$\begin{array}{ll} \log \cos 73^\circ 15' = \bar{1} \cdot 4058617. & \log \cos 73^\circ 15' = \bar{1} \cdot 4058617. \\ \log \cos 73^\circ 15' x'' = \bar{1} \cdot 4056348. & \log \cos 73^\circ 16' = \bar{1} \cdot 4053816. \\ \text{Decrease for } x'' = & 2269. \quad \text{Decrease for } 60'' = 4801. \end{array}$$

$$\begin{array}{r} \text{Hence} \quad \frac{x}{60} = \frac{2269}{4801} \\ \therefore \quad x = 28. \end{array} \quad \begin{array}{r} 2269 \\ 60 \\ 4801 \overline{) 136140} \\ 4012 \\ 171 \end{array}$$

$\therefore$ required angle = $73^\circ 15' 28''$ to the nearest second.

Note. It is important to recollect that cosine, cotangent, cosecant, and their logarithms decrease as the angle increases; consequently proportional differences must be subtracted, not added.

If the graphs of the functions are carefully drawn, it is seen that in some parts they approach much more nearly to straight lines than in others. It follows that the method of proportional parts is more accurate for some angles than for others. For a complete discussion of Proportional parts see Nixon's *Elementary Plane Trigonometry* (Clarendon Press) or any advanced textbook.

Examples IV a.

Find, from tables, the natural function of the following angles, find the logarithm of the number found, and then look up the logarithmic function in the tables. There may be a slight discrepancy in the fourth decimal place.

- | | | |
|----------------------------|----------------------------|----------------------------|
| 1. $\sin 17^\circ 15'$. | 2. $\cos 73^\circ 47'$. | 3. $\tan 16^\circ 39'$. |
| 4. $\cos 23^\circ 19'$. | 5. $\sec 67^\circ 15'$. | 6. $\cotan 44^\circ 5'$. |
| 7. $\tan 78^\circ 53'$. | 8. $\sin 83^\circ 43'$. | 9. $\cos 63^\circ 28'$. |
| 10. $\sin 156^\circ 17'$. | 11. $\tan 176^\circ 16'$. | 12. $\cot 100^\circ 10'$. |
| 13. $\cos 137^\circ 42'$. | 14. $\sin 126^\circ 37'$. | 15. $\tan 173^\circ 14'$. |

Explain carefully the difficulty that arises in connexion with some of the angles.

16. Find the Cartesian co-ordinates of a point whose polar co-ordinates are (i) $17, 16^\circ$; (ii) $25, 114^\circ$; (iii) $49, 227^\circ$.

Find the angles less than 180° which are determined by the following data:

17. $\sin \theta = .8732$.

18. $\cos A = .3469$.

19. $\sin B = .9340$.

20. $\log \tan A = \bar{1}.7932$.

21. $L \cos A = 9.7432$.

22. $\sec B = 2.5732$.

23. $\log \sin A = \bar{1}.3465$.

24. $L \tan A = 10.4385$.

25. $L \cot C = 10.7386$.

Find the sine, cosine, and tangent of the following angles, which are measured in radians:

26. $\frac{1}{2}\pi$. 27. $\frac{4}{3}\pi$. 28. 1.2 . 29. $\frac{8}{15}\pi$.

30. Verify that

$$\sin 112^\circ = \sin 70^\circ \cos 42^\circ + \cos 70^\circ \sin 42^\circ.$$

31. Find from the tables the values of

$$\sin \frac{1}{3}\pi \text{ and } \sin 27^\circ 18' / \cos 32^\circ 45'.$$

32. Employ the tables to verify the formula

$$\cot 24^\circ 45' - \cot 49^\circ 30' = \operatorname{cosec} 49^\circ 30'.$$

33. Find the values of $\cos 110^\circ$, $\cot 160^\circ$, $\sin 250^\circ$.

A quantity μ is such that $\mu = \sin i / \sin r$; complete the following tables:

	i	r	μ
34.	16°	12°	
35.	$26^\circ 18'$		1.427.
36.		$31^\circ 52'$	1.467.
37.		$53^\circ 49'$	1.5.

38. Find the polar co-ordinates of points whose Cartesian co-ordinates are (i) $(3, 7)$; (ii) $(-3, 7)$; (iii) $(-3, -7)$; (iv) $(3, -7)$.

39. The angle of friction ϵ and the coefficient of friction μ are connected by the relation $\mu = \tan \epsilon$. Determine the missing quantity in the following cases:

ϵ	$40^\circ 15'$		$17^\circ 39'$		$47^\circ 8'$		
μ		.67				.37	.50

40. In a circle of radius 17 find the lengths of chords subtending angles (i) 37° , (ii) 73° , (iii) 143° at the centre. What are the areas of the corresponding segments?

38. Graphs.

Example. Draw the graph of

$$3 \sin (x+30^{\circ})-2 \cos (x-30^{\circ}) \text{ from } x=0^{\circ} \text{ to } x=120^{\circ}.$$

In other words, draw the graph of

$$y = 3 \sin (x+30^{\circ})-2 \cos (x-30^{\circ}).$$

x	0	15°	30°	45°	60°	75°	90°	105°	120°
$\sin (x+30^{\circ})$	·500	·707	·866	·966	1·000	·966	·866	·707	·500
$\cos (x-30^{\circ})$	·866	·966	1·000	·966	·866	·707	·500	·259	0·000
$3 \sin (x+30^{\circ})$	1·500	2·121	2·598		3·000	2·898	2·598	2·121	1·500
$2 \cos (x-30^{\circ})$	1·732	1·932	2·000		1·732	1·414	1·000	·513	·000
y	-·232	·189	·598	·966	1·268	1·484	1·598	1·603	1·500

The graph is shown in Fig. XIX.

Use of Graph.

Interpolation. The value of the function can be found for any intermediate value of the angle. From the graph it is seen that $y = 1·07$ when $x = 50^{\circ}$, and $y = 1·54$ when $x = 81^{\circ}$. Calculation shows that the correct values are 1·075 and 1·542 respectively.

This is a useful method of testing the accuracy of a graph.

Maximum and Minimum. When, as x increases, y continually increases to a certain value and then decreases, that value is said to be a **maximum**; similarly, when y first decreases and then increases there is a **minimum** value. These maximum and minimum values are clearly shown on the graphs; the corresponding points are called **turning-points**.

From the graph the maximum value of

$$3 \sin (x+30^{\circ})-2 \cos (x-30^{\circ})$$

is found to be 1·62, and the corresponding angle is 99° .

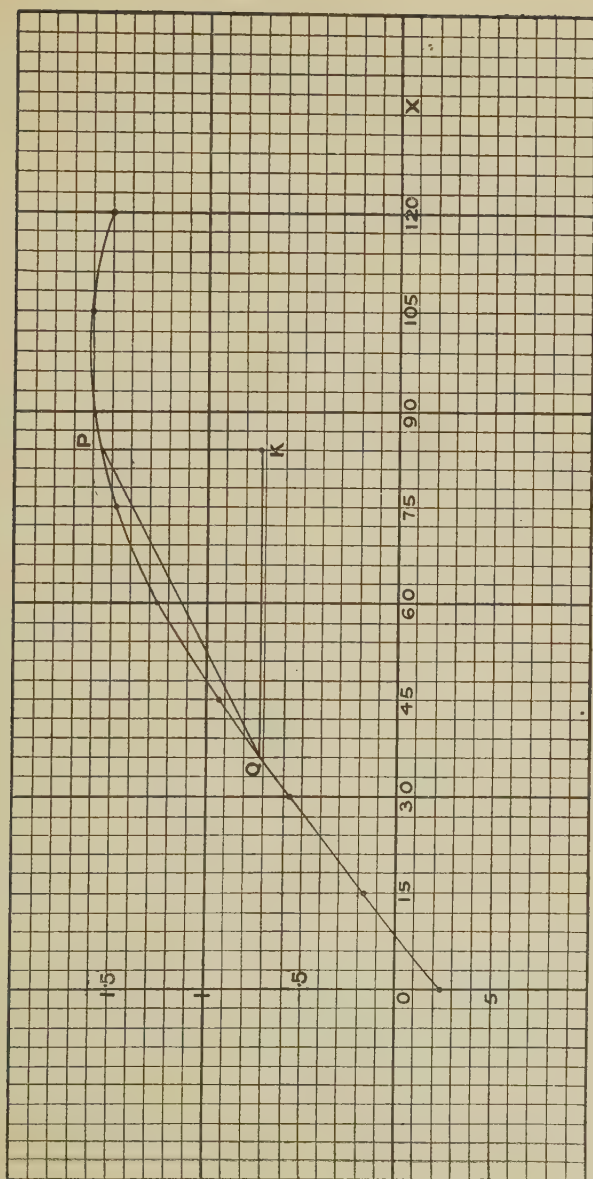


Fig. XIX.

Rate of change of the function. The graph shows that, when x changes from 15° to 30° , the increase in y is more than when x changes from 60° to 75° ; consequently the curve is steeper between 15° and 30° than between 60° and 75° . Thus the rate at which y changes compared with x is shown by the steepness of the curve.

Join two points P and Q on the graph, and draw PK , QK parallel to the axes to meet in K . Then

$$\frac{\text{increase in } y}{\text{increase in } x} = \frac{KP}{QK} = \text{tangent of the angle } PQK$$

$$= \text{tangent of the angle } PQ \text{ makes with the axis of } x.$$

This is called the **slope** of the line PQ .

When Q approaches indefinitely near to P , the chord PQ becomes a tangent. Hence

The rate of increase of y at the point P is measured by the slope of the tangent at the point P .

Notice that the slope diminishes in the neighbourhood of a turning-point and is zero at the turning-point itself.

39. Solution of equations. By finding where two graphs intersect or where one graph intersects the axis of x or a line parallel to the axis, equations can be solved just as in Algebra.

Example. Solve the equation

$$3 \sin(x + 30^\circ) - 2 \cos(x - 30^\circ) = 1.5.$$

It is seen in Fig. XIX that the graph cuts the line whose equation is $y = 1.5$ where $x = 77^\circ$ and $x = 120^\circ$. These are, therefore, the solutions within the range of the graph.

Examples IV b.

(The graphs should be verified in the way that the example of § 38 is verified.)

Draw the graphs and find the turning-points of:

1. $\sin \frac{1}{2}x$ from $x = 0^\circ$ to $x = 90^\circ$.
2. $\cos \frac{1}{2}x$ from $x = -90^\circ$ to $x = 90^\circ$.
- ③ $\sin \frac{1}{2}x + \cos \frac{1}{2}x$ from $x = 15^\circ$ to $x = 135^\circ$.
4. $\frac{1}{2} \tan(x - 60^\circ)$ from $x = 0^\circ$ to $x = 90^\circ$.
5. $\theta - \sin \theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.
6. $\sec x - \tan x$ from 0° to 90° .

7. $\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x$ from 0° to 360° .

8. Draw the graph of $\sin x + \cos x$ between $x = 0$ and $x = 360^\circ$. Solve $\sin x + \cos x = .89$, and find the slope of the graph at the points corresponding to these values of x .

9. Draw the graph of $\cos x$ between the values of 0 and 2π for x . Show that an acute angle can be found to satisfy the equation $x = \cos x$.

10. Draw the graphs from $x = -1$ to $x = +1$ of (i) $\sin^{-1}x$, (ii) $\cos^{-1}x$, (iii) $\tan^{-1}x$. How are they related to the graphs of $\sin x$, $\cos x$, $\tan x$ respectively?

11. Draw the graphs whose polar equations are

- | | |
|------------------------------|-----------------------------|
| (i) $r \sin \theta = 17$; | (ii) $r = 10 \sin \theta$; |
| (iii) $r = 10 \cos \theta$; | (iv) $\tan \theta = 2.45$. |

12. Find from your tables the values of $\cos 2x$ for the values $0^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ$ of x .

Draw the graph of $\cos 2x - \cos x$ as x increases from 0° to 60° .

13. Find, by drawing graphs of $2 \sin A$ and $\sin 2A$, for what values of A , less than 90° , $2 \sin A - \sin 2A = 1$.

14. Find, by the aid of the tables, the values of $\sin x - \tan 2x$ for the values $0^\circ, 10^\circ, 20^\circ, 45^\circ, 60^\circ$ of x .

Make a graph to give the values of $\sin x - \tan 2x$ from $x = 0$ to $x = 60^\circ$.

15. Make a table giving the values of $\cos \theta$ at intervals of one-fifth of a radian from $\theta = 0$ to $\theta =$ two radians, taking the radian as $57^\circ 30'$.

From your table plot the graph of $\theta \cos \theta$; and hence find for what value of θ , between the limits 0 and 2, $\theta \cos \theta$ is greatest.

16. Plot the function $\frac{1}{2} \{ \sin \theta + \sin 2(\theta + 20^\circ) \}$ between $\theta = 0^\circ$ and $\theta = 180^\circ$, and find the maximum and minimum values of the function which occur within this range, and the corresponding values of θ .

17. Draw, in the same diagram, the graphs of $\sin x$ and $2 \cos x$ between $x = 0^\circ$ and $x = 180^\circ$. Show how to find from your diagram an angle whose tangent is 2.

18. Taking π as 3.1416 and using your tables, find the values of $\theta - \sin \theta$ when $\theta = \frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{4}\pi, \frac{1}{5}\pi, \frac{5}{12}\pi$, and $\frac{1}{2}\pi$; and hence make a graph to give $\theta - \sin \theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

19. Draw the graph of (i) $\sin^{-1}x + \cos^{-1}x$; (ii) $\sin^{-1}(1/x)$.

Solution of Equations.

40. To solve a trigonometrical equation,

(i) express all the ratios involved in terms of one ratio,
 (ii) find the value of this ratio by ordinary algebraical methods,

(iii) find the angle from the tables,

(iv) give the general solution.

Example i. Solve $2 \sin x + 3 \cos x = 2$.

Express in terms of sine,

$$\pm 3 \sqrt{1 - \sin^2 x} = 2 - 2 \sin x.$$

Square $9 - 9 \sin^2 x = 4 - 8 \sin x + 4 \sin^2 x.$

Transpose $13 \sin^2 x - 8 \sin x - 5 = 0.$

Factorize $(13 \sin x + 5)(\sin x - 1) = 0.$

$$\therefore \sin x = -\frac{5}{13} \text{ or } 1.$$

Substituting in the original equation, we find that:

(i) If $\sin x = -\frac{5}{13}$, $\cos x = \frac{12}{13} = .9231.$

Hence the bounding line is in the fourth quadrant.

From the tables it is found that $\cos 22^\circ 37' = .9231.$

Hence the smallest positive angle satisfying the equation is

$$360^\circ - 22^\circ 37' = 337^\circ 23'.$$

But we may add or subtract any multiple of 360° without altering the position of the bounding line; hence any angle satisfies the equation whose value is $360^\circ n + 337^\circ 23'$, where n is any integer positive or negative. This is the **general solution**.

(ii) If $\sin x = 1$, $\cos x = 0.$

The smallest solution is $x = 90^\circ.$

The general solution is $360^\circ n + 90^\circ$ or $(4n + 1) 90^\circ.$

Note. The same difficulty has arisen here that arises in Algebra when the original equation contains surds. After we have squared, the resulting equation is exactly the same as if we had started with the equation $2 \sin x - 3 \cos x = 2$. For this reason, after we found the value of $\sin x$, it was necessary to substitute in the original equation to find the corresponding value of $\cos x$.

Example ii. Solve $\tan^2 A + 4 \sin^2 A = 5$.

Express in terms of $\tan A$.

$$\tan^2 A + 4 \frac{\tan^2 A}{1 + \tan^2 A} = 5.$$

Multiply by $1 + \tan^2 A$

$$\tan^4 A = 5.$$

Take logarithms

$$4 \log \tan A = .6990.$$

$$\therefore \log \tan A = .1747.$$

Use tables

$$= \log \tan 56^\circ 13'.$$

Hence a solution is

$$A = 56^\circ 13'.$$

Consideration of the fundamental figure shows that the general solution is

$$A = 180n + 56^\circ 13'.$$

41. General Solutions.

General solutions can always be obtained by mentally considering the possible positions of the radius vector that give angles having the same function as some angle already found. This is what has been done in the two preceding examples. It is, however, useful to know the formulae that give these general solutions.

Find an expression for all angles that have a sine equal to $\sin \alpha$.

We have to solve $\sin \theta = \sin \alpha$.

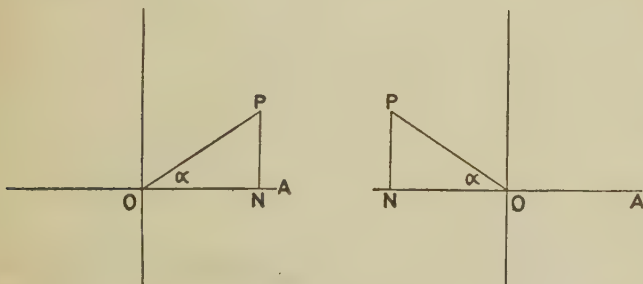


Fig. XIX a.

The bounding line may have either of the positions shown in Fig. XIX a.

Thus the line may revolve through $n\pi$, where n is even, and then go on α , or may revolve through $n\pi$, where n is odd, and then come back α . Hence $\theta = n\pi + \alpha$ if n is even,

or $n\pi - \alpha$ if n is odd.

These are included in the one formula

$$\theta = n\pi + (-1)^n \alpha.$$

If $\sin x = \sin A$, then $x = 180n + (-1)^n A$.

Exercises. In a similar way prove that

i. $\theta = 2n\pi \pm \alpha$, when $\cos \theta = \cos \alpha$;

$x = 360n \pm A$, when $\cos x = \cos A$.

ii. $\theta = n\pi + \alpha$, when $\tan \theta = \tan \alpha$;

$x = 180n + A$, when $\tan x = \tan A$.

Example. Solve $\sin 3\theta = \cos 5\theta$

This is the same as

$$\sin 3\theta = \sin \left(\frac{1}{2}\pi - 5\theta\right);$$

$$\therefore 3\theta = n\pi + (-1)^n \left(\frac{1}{2}\pi - 5\theta\right).$$

If n is odd $3\theta = n\pi - \frac{1}{2}\pi + 5\theta$;

$$\therefore 2\theta = \frac{1}{2}\pi - n\pi.$$

Put $n = -2p + 1$, then $\theta = p\pi - \frac{1}{4}\pi$, where p is any integer.

If n is even $3\theta = n\pi + \left(\frac{1}{2}\pi - 5\theta\right)$;

$$\therefore 8\theta = n\pi + \frac{1}{2}\pi.$$

Put $n = 2p$ $\theta = p\frac{1}{4}\pi + \frac{1}{8}\pi$, where p is any integer.

The complete solution is

$$\theta = p\pi - \frac{1}{4}\pi \text{ or } p\frac{1}{4}\pi + \frac{1}{8}\pi.$$

Examples IV c.

Solve:

1. $2 \cos^2 \theta = 3(1 - \sin \theta)$.

2. $\sin \theta + \cos \theta = 1$.

3. $\sin \phi + \cos \phi = \sqrt{2}$.

4. $12 \tan^2 A - 13 \tan A + 3 = 0$.

5. $2 \cos^2 x - 1 = 1 - \sin^2 x$.

6. $\sin 3A = \sin 4A$.

7. $3 \cot^4 \theta - 10 \cot^2 \theta + 3 = 0$.

8. $2 \sin A = \tan A$.

9. $\tan A + 3 \cot A = 4$.

10. $\sin(x + A) = \cos(x - A)$.

11. $\tan^2 A + 4 \sin^2 A = 6$.

12. $\sqrt{3} \tan^2 \theta + 1 = (1 + \sqrt{3}) \tan \theta$.

13. $\operatorname{cosec} \theta = \cot \theta + \sqrt{3}$.

14. $\cos(135^\circ + A) + \sin(135^\circ - A) = 0$.

15. $\cos^3 A - \cos A \sin A - \sin^3 A = 1$.

16. $\cos 3\theta + \sin \theta = 0$.

17. $3 \tan^2 2\theta = 1$.

1. $2n\pi + \frac{1}{2}\pi$ or $n\pi$
 $+ 180n^\circ + 180^\circ 26'$, $180n^\circ + 360^\circ 52'$

18. $\tan 2x = \tan 2/x$.
 19. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$.
 20. $2 \cos^2 \theta + 3 \cos \theta - 1 = 0$.
 21. $\tan^2 \theta + \sec^2 \theta = 2$.
 22. $1.7 \sin \theta - .73 = 0$.
 23. $3 \sin \theta + 2 \cos \theta = 2$.
 24. $2 \tan^2 \theta + 7 \tan \theta + 3 = 0$.
 25. $\tan \theta - 2 \cot \theta = 1.7$.

42. Examples of the use of logarithms.

Example i. Given that $\frac{a}{\sin A} = \frac{b}{\sin B}$, find B when $a = 250$, $b = 240$, $A = 72^\circ 5'$.

We have $\frac{\sin B}{b} = \frac{\sin A}{a}$,

i. e. $\sin B = \frac{b \sin A}{a}$.

Take logs. $\log \sin B = \log b + \log \sin A - \log a$
 $= 2.3802 - 2.3979$
 $+ 1.9784$
 $= 2.3586$
 $- 2.3979$
 $= 1.9607$;

[$\therefore L \sin B = 9.9607$];

$\therefore \sin B = \sin 66^\circ$.

Hence $B = 180^\circ + (-1)^n 66^\circ$.

After a little practice the work may be arranged so that the logarithms are kept quite distinct from the remainder of the work. This same example is worked below to show the shorter method and the use of five-figure tables.

$$\frac{\sin B}{b} = \frac{\sin A}{a},$$

$$\sin B = 240 \sin 72^\circ 5' \div 250$$

$$= \sin 65^\circ 59'.$$

Logarithms.

$$2.38021$$

$$+ 1.97841$$

$$2.35862$$

$$- 2.39794$$

$$\hline 1.96068$$

Example ii. *The sides and angles of a triangle are connected by the relation $\tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C$; find A and B when $a = 242.5$, $b = 164.3$, $C = 54^\circ 36'$.*

$\tan \frac{A-B}{2} = \frac{78.2}{406.8} \cot 27^\circ 18'$ $\left[= \frac{78.2}{406.8} \tan 62^\circ 42' \right]$ $= \tan 20^\circ 26';$ $\therefore \frac{A-B}{2} = 20^\circ 26',$ by question $\frac{A+B}{2} = 62^\circ 42'.$ Hence $A = 83^\circ 8',$ $B = 42^\circ 16'.$	Logarithms. $\begin{array}{r} 1.89321 \\ + .28723 \\ \hline 2.18044 \\ - 2.60938 \\ \hline 1.57106 \end{array}$
--	--

The step in brackets is required if the tables do not give the cotangents. Since A and B are angles of a triangle, $\frac{1}{2}(A-B)$ cannot equal any of the angles $180n + 20^\circ 26'$ (except when $n=0$), so that there is no need to give the general solution.

Example iii. *If a, b, c are the sides of a triangle, $\tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ where s is half the sum of the sides. Find A when $a = 1762$, $b = 893$, $c = 1386$.*

$s = (1762 + 893 + 1386) \div 2 = 4041 \div 2$ $= 2020.5,$ $s-a = 258.5,$ $s-b = 1127.5,$ $s-c = 634.5;$ $\therefore \tan \frac{A}{2} = \sqrt{\frac{1127.5 \times 634.5}{2020.5 \times 258.5}};$ $\therefore \frac{A}{2} = 49^\circ 29',$ $A = 98^\circ 58'.$	Logarithms. $\begin{array}{r} 3.05212 \\ 2.80243 \\ + 5.85455 \\ \hline 3.30546 \\ 2.41246 \\ - 5.71792 \\ \hline 2.0013663 \\ 0.06831 \end{array}$
---	--

Examples IV.

1. Use the tables to find the values of $\sin 52^\circ$, $\cos 140^\circ$, $\tan 220^\circ$, $\cos 340^\circ$, $\sin 340^\circ$.

2. Divide $\sin 52^\circ$ by $\cos 52^\circ$; verify your answer by finding the value of $\tan 52^\circ$ from the tables.

3. Write down by using tables the values of $\sin 140^\circ$, $\cos 160^\circ$, $\cos 220^\circ$, $\tan 320^\circ$.

4. Find the smallest positive value of θ which satisfies

$$\cos \theta = \sin \left\{ (4n+3) \frac{1}{2} \pi + \alpha \right\}.$$

5. Find all the values of θ which satisfy the equation

$$4 \cos \theta - 3 \sec \theta = 2 \tan \theta.$$

6. Find the inclination to the horizon of an incline which rises $5\frac{1}{2}$ feet in 420 yards.

7. Solve the equation $\tan^2 \theta - (1 + \sqrt{3}) \tan \theta + \sqrt{3} = 0$.

8. Given that $\tan \frac{1}{2} C = \sqrt{(s-a)(s-b) \div s(s-c)}$, find C when $a = 32$, $b = 40$, $c = 66$.

9. Solve the equations $\cos(2x+3y) = \frac{1}{2}$, $\cos(3x+2y) = \sqrt{3}/2$.

10. Given $\log 2 = \cdot 30103$ and $\log 3 = \cdot 47712$, find (without the tables) $L \sin 60^\circ$ and $L \tan 30^\circ$.

11. Find the acute angle whose cosine equals its tangent.

12. The current C in a circuit, as determined by a tangent galvanometer, equals $G \tan \theta$, where G is a constant depending on the galvanometer only and θ is the deflexion of the needle. Determine the ratio of two currents which give deflexions of $27^\circ 14'$, $35^\circ 23'$ respectively.

13. The length of a degree of latitude in latitude ϕ is
 $(1111 \cdot 317 - 5 \cdot 688 \cos \phi) 10^4$ centimetres.

Find the length at London (latitude $51^\circ 31' \text{N.}$) and Melbourne (latitude $37^\circ 50' \text{S.}$).

14. The length of the seconds pendulum in centimetres, at a place whose latitude is λ , is $99 \cdot 3563 - \cdot 2536 \cos 2\lambda$. Find the length of the seconds pendulum at Paris (lat. $48^\circ 50' \text{N.}$) and Calcutta (lat. $22^\circ 33' \text{N.}$).

15. The acceleration of a falling body at a place whose latitude is λ , when measured in centimetres per second per second, is

$$980 \cdot 6056 - 2 \cdot 5028 \cos 2\lambda.$$

Find the acceleration at Montreal (lat. $45^\circ 30' \text{N.}$) and Cape Town (lat. $33^\circ 40' \text{S.}$).

16. A quantity Δ is determined by the relation $\Delta = \frac{1}{2} ab \sin C$. Complete the following table:

	Δ	a	b	C
i.		17	43	$77^\circ 14'$
ii.	342.6	21.3	38.19	
iii.	984.2		43.82	$43^\circ 21'$

17. Draw the graph of

$$\tan \theta - \theta \text{ from } \theta = 0 \text{ to } \theta = \frac{\pi}{2}.$$

Hence solve $\tan \theta = \theta + 3$.

18. Given that A and B , the angles of a triangle, are connected by the relation $a \sin B = b \sin A$, find B when $a = 181$, $b = 217$, $A = 34^\circ 15'$.

19. If $2R = a/\sin A$, find the value of A when $R = 179.4$ and $a = 300$.

20. Verify that

$$\cos 146^\circ 43' - \cos 56^\circ 51' = -2 \sin 44^\circ 56' \sin 101^\circ 47'.$$

21. Find the length of (i) the chord, (ii) the arc, subtending an angle 70° at the centre of a circle of radius 25 cm. Find also the area of the segment.

22. Find the length of the side of a regular decagon (i) inscribed in, (ii) described about, a circle of radius 2.7 inches.

CHAPTER V

THE RIGHT-ANGLED TRIANGLE

43. IN the previous chapters we have had to deal with only one angle at a time, and have been able to draw one of the lines containing that angle horizontal. In applications of Trigonometry we often have to deal with several angles in the same example, and the lines containing them are drawn in various directions; in such examples it would be difficult to apply the definitions of § 12. But it has been shown that the ratios of any angle can be expressed in terms of the ratios of an acute angle. In practice, therefore, it will often be found advisable to use the following definitions, which apply only to acute angles.

In a right-angled triangle an acute angle is contained by the hypotenuse and one of the other sides which is called the side **adjacent** to that angle. The remaining side is called the side **opposite**. Then in Fig. XX

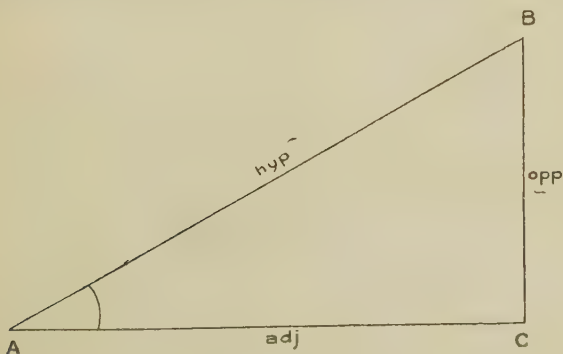


Fig. XX.

$$\sin BAC = \frac{\text{opposite}}{\text{hypotenuse}}; \quad \operatorname{cosec} BAC = \frac{\text{hypotenuse}}{\text{opposite}};$$

$$\cos BAC = \frac{\text{adjacent}}{\text{hypotenuse}}; \quad \sec BAC = \frac{\text{hypotenuse}}{\text{adjacent}};$$

$$\tan BAC = \frac{\text{opposite}}{\text{adjacent}}; \quad \cot BAC = \frac{\text{adjacent}}{\text{opposite}}.$$

These are clearly the same definitions as in § 12, the triangle BAC taking the place of the triangle PON ; and the various formulae proved in Chap. III can be proved directly from the definitions of this section.

44. It is usual to denote the angles of any triangle ABC by the capital letters A, B, C ; the lengths of the sides opposite the angles A, B, C are denoted by a, b, c respectively.

Hence, in a triangle ABC , right-angled at C ,

$$\sin A = \frac{a}{c}, \text{ i.e. } a = c \sin A;$$

$$\cos A = \frac{b}{c}, \text{ i.e. } b = c \cos A;$$

$$\tan A = \frac{a}{b}, \text{ i.e. } a = b \tan A.$$

Examples V a.

1. Prove, from the definitions of § 43, that

$$(i) \cos A = \sin B = \sin (90 - A);$$

$$(ii) \sin A = \cos B = \cos (90 - A);$$

$$(iii) \tan A = \cot B = \cot (90 - A).$$

2.

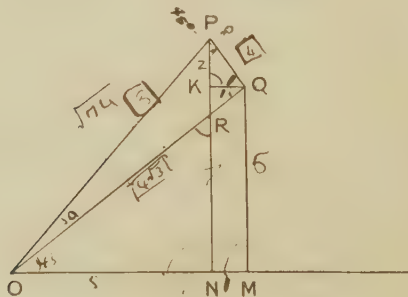


Fig. XXI.

In the figure PNO , QMO , QKP are right angles.

If $ON = 5$, $NP = 7$, $OM = 6$, $MQ = 5$, find the values of $\sin PON$, $\tan KPQ$, $\tan KQP$, $\sec QOM$, $\cos KQO$, $\operatorname{cosec} NRO$.

3. If, in Fig. XXI, $OP = 8$, $POQ = 30^\circ$, $QON = 45^\circ$, $PQO = 90^\circ$, find the lengths of OQ , PQ , PK , QM , OM .

4. A circle is described on a horizontal diameter AB of length 10 inches; a point C is taken on the circumference, such that $BC = 7$, and CD is let fall at right angles to AB . Find the size of the angle BAC and the length of CD .

5. In a triangle, right-angled at C , a perpendicular is let fall from C to the hypotenuse; prove, by Trigonometry, that this perpendicular is a mean proportional between the segments of the hypotenuse.

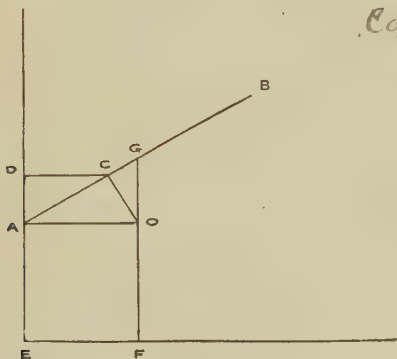


Fig. XXII.

In the above figure (which is not drawn to scale) AO is at right angles to DE , OC is at right angles to AG , OG is at right angles to AO and EF ; also G is the middle point of AB .

Use this figure in the following examples.

6. If $AC = 10$, $CAD = 40^\circ$, find, if possible, the lengths of all the other lines.

7. If $CD = 8$, $AB = 24$, find $\sin CAD$.

8. If $GF = 18$, $AE = 5$, $OC = 5$, find $\cos ACD$ and the length of AO .

$$\begin{aligned} \frac{x}{y} &= 2 \\ y &= x \cdot 2 \\ \therefore y &= 3 \end{aligned}$$



9. If $AB = l$, $CAD = \theta$, find CD and AO .

10. If $CG = a$, $CGO = \theta$, find AD and AO .

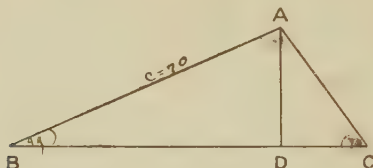


Fig. XXIII.

With the ordinary notation for the sides and angles of a triangle, find in the above figure:

11. The length of AD when $c = 70$, $B = 49^\circ$.

12. The length of AD when $b = 42$, $C = 72^\circ$.

13. The length of BD when $c = 76$, $B = 39^\circ$.

14. The length of CD when $b = 114$, $C = 114^\circ$.

(What geometrical fact does the negative sign in the result show?)

15. Prove that the area of the triangle $= \frac{1}{2}ab \sin C$; give the proof, also, when C is obtuse.

Solution of Right-angled Triangles.

45. The angles and sides of a triangle are sometimes called the six parts of a triangle. The determination of all the parts, when only some of the parts are known, is called solving the triangle. If the triangle is known to be right-angled, the triangle can be solved if one side and one other part are known.

Example i. A man, standing 100 feet from the foot of a church steeple, finds that the angle* of elevation of the top

* If a person is looking upwards, the angle his line of sight makes with the horizontal is the **angle of elevation**; similarly, if he is looking downwards, the angle his line of sight makes with the horizontal is the **angle of depression**.

is 50° . If his eye is $5\frac{1}{2}$ feet from the ground, what is the height of the steeple?

[The figure should be drawn neatly but need not be drawn to scale.]

In Fig. XXIV AE represents the steeple, BC the man; CD is drawn parallel to BA .

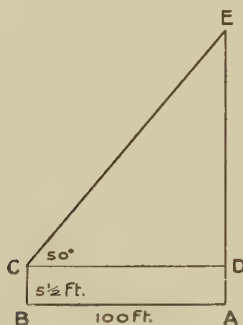


Fig. XXIV.

[Mental. In the right-angled triangle CDE we know that

$$CD = BA = 100 \text{ ft.}, \text{ angle } DCE = 50^\circ,$$

and we wish to find DE .

$$\frac{\text{unknown side}}{\text{known side}} = \text{some ratio of known angle.}]$$

Here
$$\frac{DE}{DC} = \tan DCE,$$

i.e.
$$\begin{aligned} DE &= 100 \tan 50^\circ \text{ feet} \\ &= 100 \times 1.1918 \text{ feet} \\ &= 119.18 \text{ feet.} \end{aligned}$$

Therefore height of steeple = $AD + DE = 124.68$ feet.

Example ii. *The shadow, cast by the sun on a horizontal plane, of a vertical pole 10 feet high, is observed to be 14 feet long; find the altitude of the sun (i. e. the angle of elevation of the sun).*

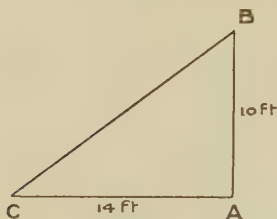


Fig. XXV.

In Fig. XXV AB represents the pole, AC the shadow; so that CB is the direction of one of the sun's rays.

[Mental. In the right-angled triangle BAC we know BA and AC , and wish to find the angle ACB .

ratio of known sides = some ratio of required angle.]

$$\frac{AB}{AC} = \tan ACB;$$

$$\therefore \tan ACB = \frac{10}{14} = \cdot 71429.$$

$$\therefore ACB = 35^{\circ} 32'.$$

$$\tan 35^{\circ} 30' = \cdot 71329.$$

440 is diff. for $10'$.

$$\therefore 100 \quad ,, \quad \frac{100'}{44} = 2'.$$

Sun's altitude = $35^{\circ} 32'$ to nearest minute.

Example iii. *A ship C is observed at the same time from two coastguard stations A and B , 1459 yards apart. The angle ABC is found to be 90° , and the angle BAC to be $67^{\circ} 14'$, what is the distance of the ship from station A ?*



$$\text{Here } \frac{AC}{AB} = \sec BAC;$$

$$\begin{aligned} \therefore \log AC &= \log 1459 + \log \sec 67^{\circ} 14' \\ &= 3 \cdot 16406 \\ &\quad + \cdot 41111 \\ &\quad + \quad 120 \\ &= 3 \cdot 57637, \end{aligned}$$

$$\text{i.e. } AC = 3770 \cdot 3.$$

Distance of ship from A = 3770 yards to nearest yard.

If the tables do not contain the secants, the working must be made to depend on the cosine.

$$\begin{aligned}\frac{AC}{AB} &= \frac{1}{\cos BAC}, \\ \log AC &= \log 1459 - \log \cos 67^\circ 14' \\ &= 3.1641 \\ &\quad - 1.5877 \\ &= 3.5764, \\ \text{i.e. } AC &= 3770.\end{aligned}$$

Example iv. Two men, *A* and *B*, 1370 yards apart, observe an aeroplane *C* at the same instant and find the respective angles of elevation to be 40° and 67° . If the plane *ABC* is vertical, calculate the height of the aeroplane.

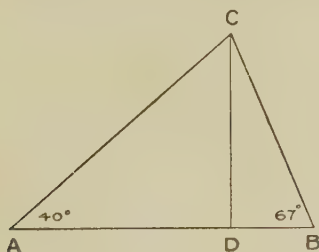


Fig. XXVII.

Let h feet be height of aeroplane.

From triangle *ADC*, $AD = h \cot 40^\circ$.

From triangle *BDC*, $BD = h \cot 67^\circ$;

but $AD + BD = AB$;

$$\therefore h \cot 40^\circ + h \cot 67^\circ = 1370.$$

$$\begin{aligned}\therefore h &= \frac{1370}{\cot 40^\circ + \cot 67^\circ} \\ &= \frac{1370}{1.61622};\end{aligned}$$

$$\therefore h = 847.62;$$

Height of aeroplane = 848 yards to nearest yard.

$\cot 40^\circ = 1.19175$
$\cot 67^\circ = .42447$
Logarithms
3.13672
$- .20852$
2.92820

Examples V b.

1. The string of a kite is known to be 500 feet long, and it is observed to make an angle of 55° with the horizontal; find the height of the kite.

2. From the top of a cliff, 215 feet high, the angle of depression of a ship is observed to be $23^{\circ} 20'$; what is the distance of the ship from the foot of the cliff?

3. From a point 56 feet from the foot of a tree the angle of elevation of the top is 73° ; find the height of the tree.

4. The top of a conical tent is 9 feet above the ground; the radius of the base is 5 feet; what is the inclination of the side of the tent to the horizontal?

5. The shadow thrown by a flagstaff is found to be $55\frac{1}{2}$ feet long when the sun's altitude is $53^{\circ} 15'$; what is the height of the flagstaff?

6. I know that a certain tower is 144 feet high. I find that its elevation observed from a certain point on the same level as the base of the tower is $37^{\circ} 16'$. Find the distance of that point from the base of the tower.

7. A sphere of radius 4 inches is suspended from a point *A* in a vertical wall so that it rests against the wall. The string is 11 inches long and is in the same straight line as a radius of the sphere. Find the inclination of the string to the vertical.

8. From the top of a cliff, 254 feet high, the angle of depression of a ship was found to be $9^{\circ} 28'$, and that of the edge of the sea $72^{\circ} 40'$; how far distant was the ship from the edge of the sea?

9. Two observers on the same side of a balloon and in the same vertical plane with it, a mile apart, find its angles of elevation to be 15° and $65^{\circ} 30'$ at the same moment. Find the height of the balloon.

10. From the top of a tower, 108 feet high, the angles of depression of the top and bottom of a vertical column are found to be 30° and 60° respectively. What is the height of the column?

11. A flagstaff, 30 feet high, is fixed in the centre of a circular tower 40 feet in diameter. From a point on the same horizontal plane as the foot of the tower the elevations of the top of the flagstaff and the top of the tower are observed to be 35° and 30° respectively. Find the height of the tower.

12. A river, the breadth of which is 200 feet, flows at the foot of a tower, which subtends an angle $25^{\circ} 10'$ at a point on the further bank exactly opposite. Find the height of the tower.

13. A person standing at the edge of a river finds that the elevation of the top of a tower on the edge of the opposite bank is 60° ; on going back 30 feet he finds the elevation to be 45° ; find the breadth of the river.

14. From the top of a tower, 50 feet high, the angle of depression of a man, walking towards the tower, is noticed to be 30° ; a few moments after it was 45° . How far had the man walked between the two observations?

15. Two posts, 400 yards apart, at the side of a straight road running E. and W., are observed to bear N. 20° E. and E. 20° N. respectively. Find the distance of the observer from the road.

16. Two points A and B and the foot D of a tower CD are in a horizontal straight line, and the angles of elevation of C , the top of the tower, as seen from A and B respectively, are $25^\circ 46'$ and $35^\circ 25'$. If the distance AB is 200 feet, find the height of the tower.

17. A vertical post casts a shadow 15 feet long when the altitude of the sun is 50° ; calculate the length of the shadow when the altitude of the sun is 32° .

18. A vertical mast, having its base at A , is set up on a horizontal plane. B and C are points in the plane in a line with A , and such that the angular elevations of the top of the mast, when observed at these points, are respectively α and β . If $\tan \alpha = \frac{3}{4}$, $\tan \beta = \frac{2}{5}$ and the length of BC is 105 feet, find the height of the mast.

19. A man standing on a tower at a height of 80 feet from the ground observes that the angles of depression of two objects on a straight level road running close to the foot of the tower are 60° and 30° . If the objects are on the same side of the tower, how far are they apart?

20. A , B , C are three points in succession on a straight level road, and P is another point so situated that the angles PAB , PBA , PCA are respectively 90° , 60° , and 45° . If a man walks at a uniform rate from A to B in 25 seconds find, to the nearest second, how long it will take him, at the same rate, to walk from B to C .

21. A ray of light passes through a hole A in a graduated horizontal scale AB in a direction perpendicular to the scale and is reflected by a vertical mirror which is distant 30 inches from the scale and makes an angle x° with the incident (i.e. approaching)

ray. After reflection the ray makes the same angle with the mirror as before and shines on the scale at a distance 8 inches from A. Find the value of x .

If the mirror now swings through an angle 1° , how far will the spot of light on the scale move?

Elementary Navigation.

{ The student should revise §§ 9 and 10 dealing with latitude and longitude and the points of the compass. }

46. When a ship is sailing, the angle between its direction of sailing and the meridian the ship is crossing is called the **course**.

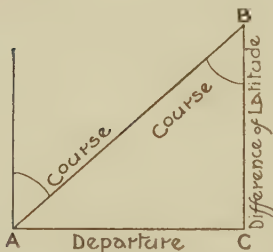


Fig. XXVIII.

If the course is constant, the ship is said to sail on a **rhumb-line**. The **distance** between two positions of the ship is then measured along the rhumb-line. The **difference of latitude** of two places is the arc of a meridian intercepted between the parallels of latitude passing through the two places. The **departure** between two meridians is the distance between the two meridians measured along a parallel of lati-

tude; thus the departure between any two given meridians is not a constant but diminishes from the equator to the poles.*

47. A small portion of the earth's surface may be regarded as a plane; for distances small, compared with the earth's radius, we may therefore use the formulae of Plane Trigonometry.

Plane Sailing is the name given to that part of navigation which treats the surface of the earth as a plane. On this assumption the meridians become parallel straight lines, the rhumb-line becomes the hypotenuse of a right-angled triangle of which the departure is the side opposite to the course, and the difference of latitude is the side adjacent. Thus problems on Plane Sailing are merely examples in the solution of right-angled triangles.

* In navigation distances are usually measured in nautical miles; a nautical mile is the length of an arc of a meridian (or the equator) which subtends an angle of $1'$ at the centre of the earth; thus a distance of 75 nautical miles is usually written $75'$.

Examples V c.

(The distances are given in nautical miles.)

1. A ship sails SE. by S., a distance 81 miles; what is her departure and difference of latitude?
2. A ship sails N. $49^{\circ} 41'$ W., a distance 73 miles; what is the departure and difference of latitude?
3. A ship sails SSW. until its departure is 198 miles; what is the distance sailed and the difference of latitude?
4. If the course is $3\frac{1}{2}$ points W. of N., and the difference of latitude 149 miles, what is the distance?
5. A ship sails between North and West, making a difference of latitude $157\frac{1}{2}$ miles and departure 79 miles; what is the course?
6. A ship sails westward 247 miles along the equator from meridian 16° E.; what is now the longitude?
7. A ship sails 247 miles eastward along the parallel 40° N.; what is the change in longitude?
8. When a ship sails any distance (great or small) along a parallel of latitude, show that
 difference of longitude in minutes = departure $\times$ secant of latitude.
9. A ship, from latitude $54^{\circ} 22' 10''$ N., sails $195\frac{1}{2}$ miles $\frac{1}{4}$ of a point S. of SE.; what is now the latitude?
10. Leaving latitude $49^{\circ} 37'$ N., longitude $15^{\circ} 22'$ W., a ship sails SW. by W. 150 miles; find the new latitude and longitude.

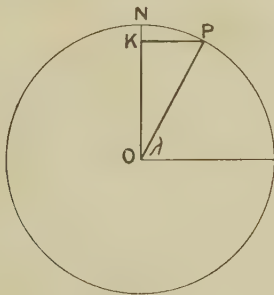


Fig. XXIX.

48. Parallel Sailing. If λ° is the latitude, then the radius of the parallel of latitude (KP in Fig. XXIX) is $\cos \lambda \times$ radius of the

earth. If θ is the radian measure of the difference of longitude of two places on the same parallel, the length of the arc between them is $\theta \cos \lambda \times$ radius of the earth. The radius of the earth is $\frac{21600}{2\pi}$ nautical miles.

$$\text{Hence} \quad \text{departure} = \frac{1600}{2\pi} \theta \cos \lambda.$$

When θ is reduced to minutes, this relation becomes

$$\text{departure} = \text{difference of longitude} \times \text{cosine of latitude.}$$

49. Middle Latitude Sailing. In Middle Latitude Sailing, the departure between two places, whose latitudes are λ and λ' , is taken to be the departure between their meridians, measured at the latitude $\frac{1}{2}(\lambda + \lambda')$. On this assumption,

$$\text{departure} = \text{diff. of longitude} \times \cos \frac{1}{2}(\lambda + \lambda').$$

50. Traverse Sailing. If a ship sails on different courses, from A to B , from B to C , from C to D , &c., then, by the methods of Plane Sailing, the total changes in latitude and longitude can be worked out. This is called the method of **Traverse Sailing**. This method can only be used when the whole area traversed can be regarded as plane without introducing a great amount of error.

Example. *A ship left a position in which Oporto Light (lat. $41^\circ 9' N.$, long. $8^\circ 38' W.$) bore $W.$ by $N.$, 15 miles distant. Afterwards she sailed as under :*

Courses.	Distances.
N.E.	70'
S. by W. $\frac{1}{2}$ W.	55'
E.	35'
N.N.W.	42'
S.W.	51'.

Find her bearing and distance from the Light in her last position.

We have a series of right-angled triangles to solve, the hypotenuse and an acute angle being given in each case. In practical navigation special tables are used, called Traverse Tables.

The angle the hypotenuse makes with the meridian is taken in each case.

O to *A*.

Hypotenuse 15', angle 7 points = $78^{\circ} 45'$.	1.1761
Diff. of latitude = $15' \times \cos 78^{\circ} 45'$	<u>1.2902</u>
= 2.926' S.	.4663
Departure = $15' \times \sin 78^{\circ} 45'$	1.1761
= 14.71' E.	<u>1.9916</u>
	1.1677

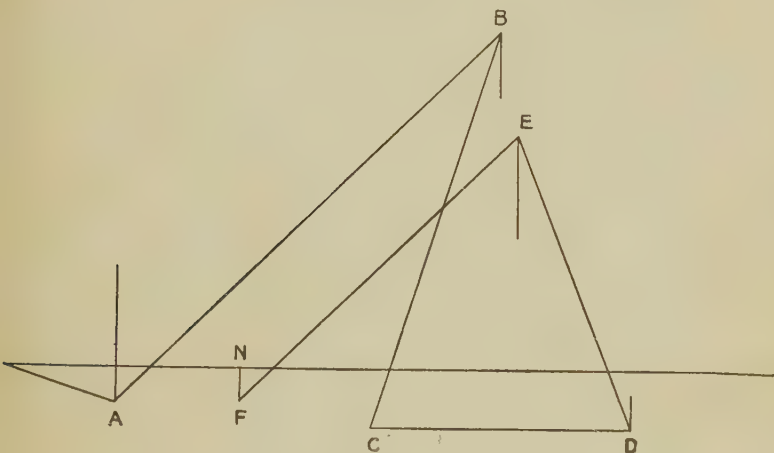


Fig. XXX.

Note that *O* bears W. by N. from *A*, but *A* bears E. by S. from *O*.

A to *B*.

Hypotenuse 70', angle 45° .	.7071
Diff. of latitude = $70 \times \cos 45^{\circ}$	
= 49.497' N.	
Departure = 49.497' E.	

B to *C*.

Hypotenuse 55', angle $1\frac{1}{2}$ points = $16^{\circ} 52\frac{1}{2}'$.	1.7404
Diff. of latitude = $55 \cos 16^{\circ} 52\frac{1}{2}'$	<u>1.9809</u>
= 52.64' S.	1.7213
Departure = $55 \sin 16^{\circ} 52\frac{1}{2}'$	1.7404
= 15.97' W.	<u>1.4628</u>
	1.2032

The other triangles are worked in the same way.

Tabulate the results thus :

Direction.	Distance.	Diff. of Latitude.		Departure.	
		N.	S.	E.	W.
E. by S.	15'		2° 33'	14° 71'	
NE.	70'	49° 50'		49° 50'	
S. by W. $\frac{1}{2}$ W.	55'		52° 64'		15° 97'
E.	35'			35	
NNW.	42'	38° 80'			16° 07'
S. W.	51'		86° 06'		36° 06'
		88° 30'	91° 63'	99° 21'	68° 10'
			88° 30'	68° 10'	
			3° 33'	31° 11'	

We see now that the final difference of latitude from the light is 3° 33' S., and departure 31° 11' E.; so that we have to solve a right-angled triangle given the two sides.

$$\begin{array}{rcl}
 \text{In Fig. XXX} & \tan FON = \frac{3'33}{31'11}; & \begin{array}{r} .5224 \\ 1'4929 \end{array} \\
 & \therefore FON = 6^\circ 7'. & \begin{array}{r} 1'0295 \\ 1'4929 \end{array} \\
 & OF = \frac{31'11}{\cos FON} & \begin{array}{r} 1'9975 \\ 1'4954 \end{array} \\
 & = 31'29 &
 \end{array}$$

In her final position the ship bore 6° 7' S. of E., 31·3 miles distant from the Light.

To find the longitude of the ship.

Latitude of <i>O</i>	41° 9' N.
Diff. of latitude for <i>F</i>	3° 3' S.
Latitude of <i>F</i>	41° 5' 7" N.
Middle latitude	41° 7'.

$$\begin{array}{rcl}
 \text{Difference of longitude in minutes} & = & \frac{\text{departure}}{\cosine \text{ of middle latitude}} \\
 & & \begin{array}{r} 31'11 \quad 1'4929 \\ = \cos 41^\circ 7' \quad -1'8770 \\ = 41'30'. \quad 1'6159 \end{array}
 \end{array}$$

$$\begin{array}{l}
 \text{Longitude of } F = 8^\circ 38' - 41'30' \\
 = 7^\circ 57' \text{ W.}
 \end{array}$$

Examples V d.

1. Find the distance on the parallel between Cape Agulhas (lat. $34^{\circ} 50' S.$, long. $20^{\circ} 1' E.$) and Monte Video (lat. $34^{\circ} 50' S.$, long. $56^{\circ} 9' W.$).

2. A ship steamed at the rate of 12 knots from Albany (lat. $35^{\circ} 3' S.$, long. $118^{\circ} 2' E.$) to Cape Catastrophe (lat. $35^{\circ} 3' S.$, long. $135^{\circ} 58' E.$). How long did she take on the voyage?

3. A ship sailed from Port Elizabeth (lat. $34^{\circ} 7' S.$, long. $25^{\circ} 40' E.$) SE. $\frac{1}{2} S.$, until her departure was 397'; find her final position.

4. Find the course and distance from Syracuse (lat. $37^{\circ} 3' N.$, long. $15^{\circ} 15' E.$) to Fano (lat. $39^{\circ} 52' N.$, long. $19^{\circ} 19' E.$).

5. A ship left a position from which Cape Clear (lat. $51^{\circ} 26' N.$, long. $9^{\circ} 29' W.$) bore NE. by E. 12.5 miles distant and sailed South 150' and then West 290 miles. Find the bearing and distance of Cape Clear from the ship in her last position.

6. Find, by Middle Latitude Sailing, the departure between two places whose positions are $13^{\circ} S.$, $50^{\circ} E.$ and $20^{\circ} S.$, $60^{\circ} E.$

7. A ship sails from $50^{\circ} N.$, $50^{\circ} W.$ to latitude $48^{\circ} N.$, the distance being 157'; find the new longitude.

8. Cape Ortegal (lat. $43^{\circ} 45' N.$, long. $7^{\circ} 6' W.$) bore SW. $\frac{3}{4} W.$ 12 miles distant. Afterwards sailed as under:

True Courses.	Distances.
NNW. $\frac{1}{2} W.$	70'
ESE.	85'
NNE. $\frac{3}{4} E.$	101'
S.	50'
WSW.	92'



Find the final latitude and longitude.

9. A ship left the Texel (latitude $52^{\circ} 58' N.$) and then sailed W. by N. $34'$, S. by E. $45'$, W. by S. $35'$, SSE. $44'$, WSW. $\frac{1}{2} W.$ $42'$. Find the course and distance to Dungeness which lies 139' West of the Texel in latitude $50^{\circ} 55' N.$

10. A ship, latitude $17^{\circ} 10' N.$, is making for a harbour, latitude $13^{\circ} 10' N.$, and $180'$ W. of the ship. She sails SW. by W. $27'$, WSW. $\frac{1}{2} W.$ $30'$, W. by S. $25'$, W. by N. $18'$, SSE. $32'$, SSE. $\frac{3}{4} E.$ $27'$, S. by E. $25'$, S. $31'$, SSE. $39'$. Find the course and distance to the harbour.

11. A ship left a position in which Heligoland bore ENE. $12'$, and then sailed NW. $24'$, S. by W. $20'$, NW. by W. $32'$, S. by E. $36'$, WNW. $\frac{1}{2}$ W. $42'$, SSE. $\frac{1}{2}$ E. $16'$, W. $\frac{3}{4}$ N. $45'$. What is then the position of the ship? Heligoland lies $54^{\circ} 12' \text{ N.}$, $7^{\circ} 54' \text{ E.}$

12. A ship sailed from Barcelona ($41^{\circ} 25' \text{ N.}$, $2^{\circ} 10' \text{ E.}$) SE. by E. $\frac{1}{2}$ E. until she reached latitude $36^{\circ} 21' \text{ N.}$ What was then her longitude?

13. A ship left a position in which Sable Island ($43^{\circ} 24' \text{ N.}$, $65^{\circ} 36' \text{ W.}$) bore NW. $\frac{1}{2}$ W., distant 12 miles.

Afterwards sailed as under :

Courses.	Distances.
ESE.	$72'$
SW. $\frac{1}{4}$ W.	$37'$
NNE.	$42'$
E.	$25'$

Required the latitude and longitude reached.

51. The Double Angle.

In Fig. XXXI, the angle $BAC = A^{\circ}$; on AB a semicircle is described with centre O , so that angle $BOC = 2A$.

Let fall CN perpendicular to AB .

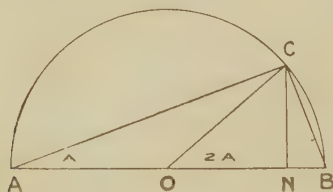


Fig. XXXI.

$$\begin{aligned}
 \cos 2A &= \frac{ON}{OC} \\
 &= \frac{AN - AO}{OC} \\
 &= \frac{AN}{OC} - 1
 \end{aligned}$$

$$\left\{ = \frac{2AN}{AC} \cdot \frac{AC}{2OC} - 1 \right. \quad \left. \begin{array}{l} \text{Fill in the vacant places with the} \\ \text{hypotenuse of the triangle of} \\ \text{which } AN \text{ is a side.} \end{array} \right\}$$

$$= 2 \frac{AN}{AC} \cdot \frac{AC}{AB} - 1$$

$$= 2 \cos^2 A - 1.$$

$$2 - 2 \sin^2 A$$

Exercises. In a similar way prove

i. $\sin 2A = 2 \sin A \cos A.$

ii. $\cos 2A = 1 - 2 \sin^2 A.$

Deduce

iii. $\cos 2A = \cos^2 A - \sin^2 A.$

iv. $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$

v. $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$; $\cos A = \cos^2 \frac{1}{2} A - \sin^2 \frac{1}{2} A.$

vi. $2 \cos^2 \frac{1}{2} A = 1 + \cos A.$

vii. $2 \sin^2 \frac{1}{2} A = 1 - \cos A.$

viii. $\tan \frac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}.$

ix. Prove the formulae for $\sin 2A$ and $\cos 2A$ when $2A$ is obtuse.

x. Do these proofs apply to angles of any size? If not; between what limits do they apply? Why is the ambiguous sign omitted in viii?

52. Geometrical questions may often be solved by using Trigonometry. For example:

If from a point outside a circle a secant and a tangent be drawn, the rectangle contained by the whole secant and the part outside the circle is equal to the square on the tangent.

In Fig. XXXII it is required to prove that $\text{rect. } PA \cdot PB = \text{sq. on } PT$.

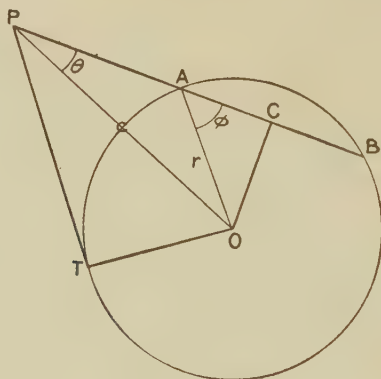


Fig. XXXII.

Let radius = r , $OP = c$, and angle $OPB = \theta$, angle $OAC = \phi$.

$$PA = PC - AC$$

$$= c \cos \theta - r \cos \phi.$$

$$PB = PC + CB$$

$$= PC + AC \quad (\text{Prop. 21})$$

$$= c \cos \theta + r \cos \phi$$

$$\therefore PA \cdot PB = c^2 \cos^2 \theta - r^2 \cos^2 \phi$$

$$= c^2 - r^2 - c^2 \sin^2 \theta + r^2 \sin^2 \phi.$$

But $c \sin \theta = OC$ from triangle OPC

$$= r \sin \phi \text{ from triangle } OAC.$$

Hence $PA \cdot PB = c^2 - r^2$

$$= OP^2 - OT^2$$

$$= PT^2 \text{ since } OTP \text{ is a right angle.}$$

53. Known results in Geometry are useful for proving Trigonometrical relations.

Show that, in any triangle,

$$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

With centre C and radius CA (i.e. b), describe a circle cutting CB in E and CB produced in D .

Then $BE = a - b$, and $BD = a + b$.

Join AD and AE .

Through E draw EF parallel to DA and meeting AB at F .

Then the angle DCA at the centre $= 180 - C = A + B$.

So that the angle DEA at the circumference $= \frac{1}{2}(A + B)$.

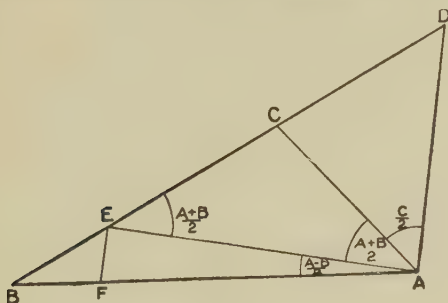


Fig. XXXIII.

Also the angle $BAE = BAC - EAC = A - \frac{1}{2}(A + B) = \frac{1}{2}(A - B)$.

Also the angle EAD , being in a semicircle, is a right angle.

$$\tan \frac{1}{2}(A + B) = \frac{AD}{AE},$$

$$\tan \frac{1}{2}(A - B) = \frac{EF}{AE}, \text{ since } AEF = EAD = \text{a right angle.}$$

Hence

$$\begin{aligned} \frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} &= \frac{EF}{AD} \\ &= \frac{BE}{BD} \text{ since } EF \text{ is parallel to } AD. \text{ (Prop. 12 a.)} \\ &= \frac{a - b}{a + b}. \end{aligned}$$

Corollary. $A + B + C = 180^\circ$; $\therefore \frac{1}{2}(A + B) = 90 - \frac{1}{2}C$.

Hence the above result may be written

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \frac{1}{2}C.$$

This formula will be used in a later chapter.

Examples V.

In the following examples:

A, B, C are the angles of a triangle ABC .

a, b, c are the sides, s = half the sum of the sides; R is the radius of the circumcircle.

r is the radius of the inscribed circle.

r_1 is the radius of the escribed circle touching the side BC .

Δ is the area of the triangle.

D, E, F are the middle points of the sides BC, CA, AB , respectively.

X, Y, Z are the feet of the perpendiculars let fall from A, B, C respectively on the opposite sides.

O is the centre of the circumcircle.

I is the centre of the inscribed circle.

K is the orthocentre.

1. Express in terms of the sides and angles the lengths of AX, BX, CX, AK, BK, CK .

2. Express the length of AD in terms of (i) a, b, C , (ii) a, b, B , (iii) a, b, c .

3. Show that $a/\sin A = b/\sin B = c/\sin C = 2R$. Deduce that $R = abc/4\Delta$.

4. Prove that $r(\cot \frac{1}{2}B + \cot \frac{1}{2}C) = a$. Write down the two similar formulae.

5. Prove that $r = \Delta/s$. (No trigonometry required.)

Deduce that $\tan \frac{1}{2}A = \Delta \div \{s(s-a)\}$.

6. Show that $BX = a - b \cos C$; hence prove that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

7. Prove that (i) $\Delta = \frac{1}{2}ab \sin C$, (ii) $\Delta = rs$, (iii) $\Delta = abc \div 4R$, (iv) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

8. Prove that

$$\begin{aligned} \text{(i)} \quad \sin \frac{1}{2}A &= \sqrt{(s-b)(s-c) \div bc}, \quad \text{(ii)} \quad \cos \frac{1}{2}A = \sqrt{s(s-a) \div bc}, \\ \text{(iii)} \quad \tan \frac{1}{2}A &= \sqrt{(s-b)(s-c) \div s(s-a)}. \end{aligned}$$

9. Show that the triangles ABC and AYZ are equiangular; hence prove that $YZ = a \cos A$.

10. Two tangents are drawn from a point P to a circle of radius 10 cm.; the tangents contain an angle of 43° . Find the lengths of the tangents and the distance of P from the centre.

11. A sheet of iron is shaped so that it can be rolled up to form a conical funnel 6 feet high with open circular ends 2 feet and 6 feet diameter respectively. Draw a plan of the sheet before rolling. What is the inclination of the edge of the funnel to the line joining the centres of the ends?

12. A circle rolls without slipping along a straight line: prove that the co-ordinates of a point fixed to the circumference are such that $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$; the origin being taken at the point where the fixed point meets the straight line, and θ being the angle turned through by the circle.

13. One of the angles of a right-angled triangle is the acute angle whose sine is $\frac{2}{3}$, and the length of the shortest side of the triangle is 10 feet. Find the lengths of the other two sides.

14. A is the highest point of a sphere with centre O ; a particle slides from a position P , where the angle $AOP = \theta$, to the position Q where the angle AOQ is ϕ . How much lower is Q than P and how much further from OA ?

15. The time t of sliding from rest down a length s inclined at θ to the horizon is given by $s = \frac{1}{2}gt^2 \sin \theta$ where g is a constant. A circle is held with a diameter AB vertical; prove that the time of sliding along a chord from the highest point A to the circumference is the same whatever be the inclination of the chord, and that the time of sliding from the circumference along a chord to B is also independent of the inclination of the path.

16. A plane, inclined at 20° to the horizon, is placed with the line of greatest slope pointing north. A line is drawn on the plane, pointing NNE.; find the inclination of this line to the horizontal.

17. A man 6 feet high walks along a straight line which passes 3 feet from a lamp-post. If the light is 9 feet from the ground, find the length of the man's shadow when his distance from the point on his path nearest to the lamp is 10 feet. What is the locus traced out by the extremity of his shadow as he walks along the line?

18. If, in the previous question, there is a vertical wall parallel to the man's path and distant 2 feet from it on the side remote from the lamp, what is then the length of the shadow and the locus traced by its extremity?

19. Draw the graph of $\theta/\sin \theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

Use the graph to solve the following problem.

A string 30 inches long is tied to the ends of a cane 35 inches long, thus forcing the cane into a circular arc. Find the radius of the arc correct to the nearest inch.

20. Find the length of a strap which passes tightly round two pulleys of radii 2 feet and 3 feet, their centres being 6 feet apart.

CHAPTER VI

THE TRIANGLE

SEVERAL formulae connecting the sides and angles of a triangle have been proved in the examples of the preceding chapters. They are here gathered together for reference and proofs are given. Care should be taken that the proof applies when the triangle is obtuse-angled; if it does not, a separate proof must be given.

Relations between the sides and angles.

54. The angle formula. $A + B + C = 2$ right angles.

The sine formula. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} (= 2 R).$

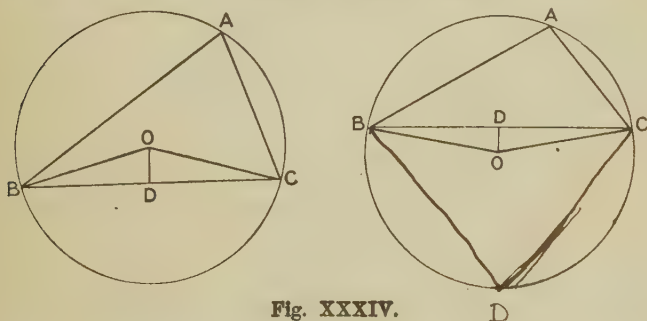


Fig. XXXIV.

Let O be the centre of the circumcircle, and D the middle point of BC .

Join OB , OC , OD .

Then, in the left-hand circle of Fig. XXXIV,

$$\begin{aligned} \text{angle } BOC &= 2^{\text{ce}} \text{ angle } BAC \\ &= 2A. \end{aligned}$$

Triangles BOD and COD are congruent;

(Prop. 8a.)

$$\therefore BOD = COD = A.$$

Also

$$BD = \frac{1}{2} BC = \frac{1}{2} a.$$

In the right-angled triangle BOD ,

$$BD = OB \sin BOD,$$

$$\text{i. e. } \frac{1}{2}a = R \sin A;$$

$$\therefore \frac{a}{\sin A} = 2R.$$

In a similar way it may be proved that

$$\frac{b}{\sin B} = 2R \text{ and } \frac{c}{\sin C} = 2R.$$

Hence

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Exercise. Supply the proof when the angle A is obtuse.

Note. In using this formula the following algebraic result is often useful :

$$\text{If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \text{ then each fraction equals } \frac{pa + qc + re}{pb + qd + rf}.$$

55. The cosine formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, and its equivalent $a^2 = b^2 + c^2 - 2bc \cos A$.

This can be proved very shortly by assuming Euclid II, 12 and 13; but it is better to base the proof on the theorem of Pythagoras.

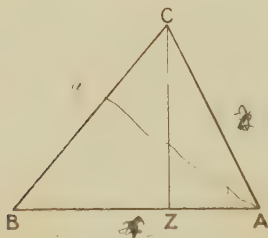


Fig. XXXV.

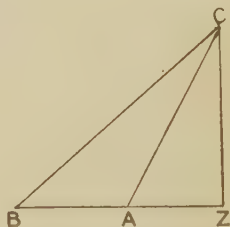


Fig. XXXV a.

Let CZ be the perpendicular from C on AB , Fig. XXXV. Then $ZC = b \sin A$, $AZ = b \cos A$, and $BZ = c - b \cos A$.

$$BC^2 = BZ^2 + ZC^2,$$

$$a^2 = (c - b \cos A)^2 + (b \sin A)^2$$

$$= c^2 - 2bc \cos A + b^2 \cos^2 A + b^2 \sin^2 A,$$

$$\text{i. e. } a^2 = b^2 + c^2 - 2bc \cos A,$$

$$\text{or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

If A is obtuse, then in Fig. XXXV a,

$$ZC = b \sin (180 - A) = b \sin A,$$

$$AZ = b \cos (180 - A) = -b \cos A,$$

$$BZ = BA + AZ = c + (-b \cos A) = c - b \cos A.$$

The proof is now the same as before.

Exercise. Write down the corresponding formulae for $\cos B$ and $\cos C$.

56. The Projection formulae

$$c = b \cos A + a \cos B.$$

In Fig. XXXV, BZ is the projection of BC on BA ; and AZ is the projection of AC .

$$AB = AZ + BZ,$$

i.e. $c = b \cos A + a \cos B.$

Exercises. Supply the proof when A is obtuse.

Write down the other two corresponding formulae.

57. Area formulae

The symbol Δ is used to denote area of triangle.

(i) $\Delta = \frac{1}{2}$ any side $\times$ perpendicular from opposite angle.

(Prop. 16.)

(ii) $\Delta = \frac{1}{2} AB \times ZC = \frac{1}{2} cb \sin A = \frac{1}{2} bc \sin A.$

(iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$

In Fig. XXXV, $BZ = a \cos B = a \times \frac{c^2 + a^2 - b^2}{2ca}.$

$$ZC^2 = a^2 - \frac{(c^2 + a^2 - b^2)^2}{(2c)^2};$$

$$\begin{aligned} \therefore (2c \cdot ZC)^2 &= (2ac)^2 - (c^2 + a^2 - b^2)^2 \\ &= (a^2 + 2ac + c^2 - b^2)(b^2 - a^2 - 2ac + c^2) \\ &= (a+b+c)(a-b+c)(a+b-c)(b+c-a). \end{aligned}$$

Let $2s = a+b+c$, then $b+c-a = 2(s-a)$ &c.; so that

$$2c \cdot ZC = \sqrt{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)};$$

$$\begin{aligned} \therefore \Delta &= \frac{1}{2} AB \cdot ZC \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

Exercise. Show that

$$16 \Delta^2 = 2(b^2c^2 + c^2a^2 + a^2b^2) - (a^4 + b^4 + c^4).$$

58. From these formulae others may be deduced.

Example i. *To show that in any triangle*

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

From sine formula $a \sin B - b \sin A = 0.$ (i)

From projection formula $a \cos B + b \cos A = c.$ (ii)

Square and add, $a^2 + b^2 + 2ab(\cos A \cos B - \sin A \sin B) = c^2.$

From cosine formula $a^2 + b^2 - 2ab \cos C = c^2.$

It follows that

$$\cos C = -(\cos A \cos B - \sin A \sin B).$$

From the angle formula $C = 180 - (A+B),$

i.e. $\cos C = -\cos(A+B).$

Hence $\cos(A+B) = \cos A \cos B - \sin A \sin B.$

Example ii. *In any triangle*

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

Multiply together equations (i) and (ii) above.

$$a^2 \sin B \cos B - b^2 \sin A \cos A - ab(\sin A \cos B - \cos A \sin B) = 0.$$

From Fig. XXXVI it is seen that

$$a^2 \sin B \cos B = BZ \cdot ZC = 2^{\text{ce}} \text{ triangle } BZC,$$

$$\text{and } b^2 \sin A \cos A = 2^{\text{ce}} \text{ triangle } AZC$$

$$= 2^{\text{ce}} \text{ triangle } A'ZC,$$

($ZA' = ZA$, so that triangles CZA, CZA' are congruent).

$$\begin{aligned} \therefore a^2 \sin B \cos B - b^2 \sin A \cos A &= 2^{\text{ce}} \text{ triangle } BCA' \\ &= BC \cdot CA' \sin BCA' \\ &= ab \sin(A-B). \end{aligned}$$

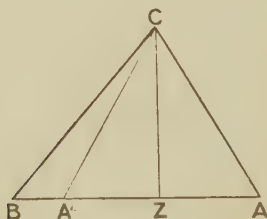


Fig. XXXVI.

Comparing this with the result above, we see that

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

This result can, however, be obtained more quickly.

For
$$\frac{\sin BCA'}{A'B} = \frac{\sin CBA'}{CA'},$$

i.e.
$$\frac{\sin(A-B)}{a \cos B - b \cos A} = \frac{\sin B}{b}.$$

Hence
$$\sin(A-B) = \frac{a \sin B \cos B}{b} - \cos A \sin B$$

$$= \sin A \cos B - \cos A \sin B$$

since $a \sin B = b \sin A$

Example iii. To show that the area of a quadrilateral inscribed in a circle is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s = \frac{1}{2}(a+b+c+d)$.

In Fig. XXXVII

Area of $ABCD$ = sum of triangles ABD and BCD

$$= ad \sin A + \frac{1}{2} bc \sin(180-A)$$

$$= \frac{1}{2} (ad + bc) \sin A.$$

From triangle ABD ,

$$BD^2 = a^2 + d^2 - 2ad \cos A.$$

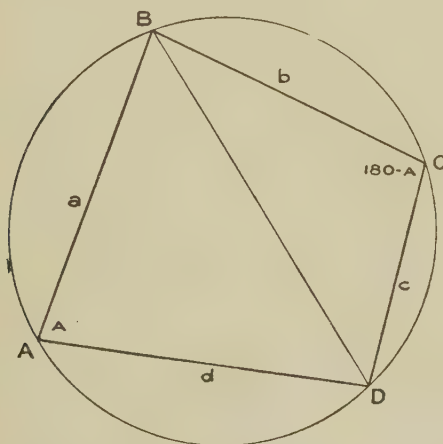


Fig. XXXVII.

From triangle BCD ,

$$BD^2 = b^2 + c^2 - 2bc \cos(180-A).$$

Hence
$$a^2 + d^2 - 2ad \cos A = b^2 + c^2 + 2bc \cos A,$$

i.e.
$$2(ad + bc) \cos A = a^2 + d^2 - (b^2 + c^2);$$

$\therefore 2(ad + bc)(1 + \cos A) = (a + d)^2 - (b - c)^2,$

and
$$2(ad + bc)(1 - \cos A) = (b + c)^2 - (a - d)^2.$$

$$\begin{aligned}
 &\text{Hence } 4(ad+bc)^2(1-\cos^2 A) \\
 &\quad = (-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d), \\
 \text{i.e. } &\left\{\frac{1}{2}(ad+bc)\sin A\right\}^2 \\
 &= \frac{1}{2}(-a+b+c+d)\frac{1}{2}(a-b+c+d)\frac{1}{2}(a+b-c+d)\frac{1}{2}(a+b+c-d) \\
 \therefore &\text{Area of } ABCD = \sqrt{(s-a)(s-b)(s-c)(s-d)}.
 \end{aligned}$$

Examples VIa.

1. From the three projection formulae deduce the three cosine formulae.

2. Prove that $\sin A = \sin B \cos C + \cos B \sin C$; and deduce that $\sin(B+C) = \sin B \cos C + \cos B \sin C$.

3. Prove that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

4. Show that $\Delta = \frac{1}{2}(b^2 \sin C \cos C + c^2 \sin B \cos B)$.

5. Show that $\Delta = \frac{1}{2}c^2 \{\sin A \sin B \div \sin(A+B)\}$.

6. Prove that $\sin A + \sin B > \sin C$.

7. Prove that $\cot A + \cot B = c \operatorname{cosec} B \div a$.

What third expression are these equal to?

8. Show that

R (i.e. the radius of the circumcircle) $= s \div (\sin A + \sin B + \sin C)$.

9. Use the formula $\cos A = 1 - 2 \sin^2 \frac{1}{2} A$ to prove that

$$\sin \frac{1}{2} A = \sqrt{(s-b)(s-c) \div bc}.$$

Write down the similar formulae for $\sin \frac{1}{2} B$ and $\sin \frac{1}{2} C$.

10. In a similar way to that suggested in the previous example, prove that $\cos \frac{1}{2} A = \sqrt{s(s-a) \div bc}$. Write down the formulae for $\cos \frac{1}{2} B$ and $\cos \frac{1}{2} C$. What is the formula for $\tan \frac{1}{2} A$?

11. Given $a = 17$, $b = 12$, $B = 37^\circ 15'$, find A .

12. Given $a = 14$, $b = 13$, $c = 12$, find the greatest angle.

13. Given $a = 45$, $A = 45^\circ$, $B = 60^\circ$, find b .

14. Given $b = 17$, $c = 42$, $A = 72^\circ$, find a .

15. Given $a = 176$, $b = 291$, $c = 352$, find all the angles.
(Choose a formula adapted for logarithms.)

16. Given $a = 7$, $b = 5$, $C = 49^\circ$, find c .

17. Given $b = 9$, $c = 10$, $C = 57^\circ$, find a .

18. By considering two forms for the area of an isosceles triangle, prove that $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$.

19. Two sides of a triangle are 3 and 12 and the contained angle is 30° ; find the hypotenuse of an isosceles right-angled triangle of equal area.

20. Two adjacent sides of a parallelogram, 5 inches and 8 inches long respectively, include an angle of 60° . Find the length of the two diagonals and the area of the figure.

21. If in a triangle $C = 60^\circ$, prove that

$$1/(a+c) + 1/(b+c) = 3/(a+b+c).$$

22. On a straight line AB , 4 inches long, describe a semicircle, and on the arc of the semicircle find points P, Q, R, S such that the areas of the triangles APB, AQB, ARB, ASB are 1 square inch, 2 square inches, 3 square inches, and 4 square inches respectively. If C is the centre of the circle, determine the sines of the angles ACP, ACQ, ACR , and ACS , and hence find, from the tables, the values of these angles.

23. If a quadrilateral can be inscribed in one circle and circumscribed about another, show that its area is $\sqrt{abcd}$, where a, b, c, d are the lengths of the sides.

The circles of the triangle.

59. It is shown in any Geometry textbook that

(i) the centre of the circumcircle is the point of concurrence of the perpendiculars drawn at the middle points of the sides;

(ii) the centre of the inscribed circle is the point of concurrence of the three lines bisecting the three angles;

(iii) the centre of an escribed circle is the point of concurrence of the bisector of the opposite interior angle with the bisectors of the two adjacent exterior angles.

In Fig. XXXVIII, we have

$$AQ = AR, \quad (\text{Prop. 24.})$$

$$BP = BR,$$

$$CP = CQ;$$

$$\therefore AQ + BP + CP = \frac{1}{2} \text{ sum of sides} = s.$$

Hence

$$AQ = s - a.$$

Exercise. In a similar way, prove that

$$BP = \quad \quad CP' =$$

$$CQ = \quad \quad QQ' =$$

$$AQ' = \quad \quad PP' =$$

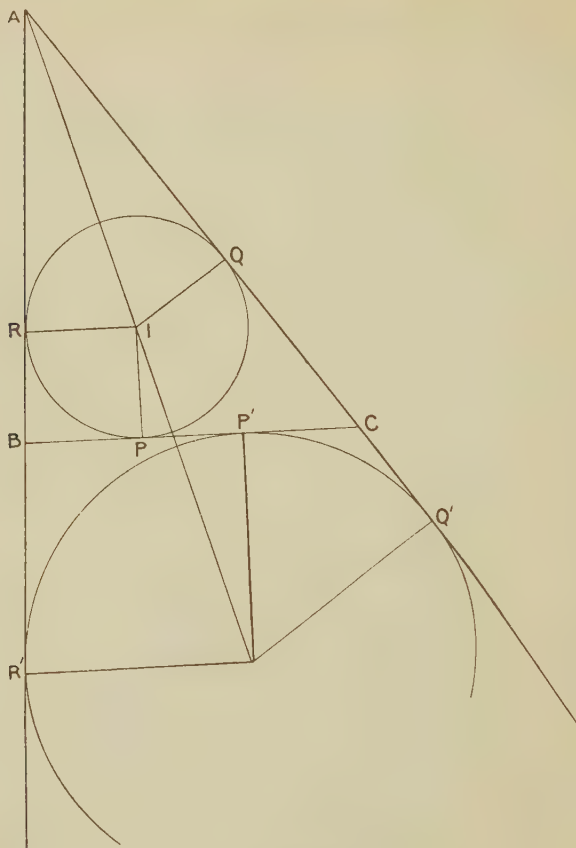


Fig. XXXVIII.

Examples VI b.

Prove the following formulae :

1. $R = a \div 2 \sin A.$
2. $R = abc \div 4 \Delta.$
3. $r = \Delta/s.$ (Consider the sum of the triangles BIC , CIA , AIB .)
4. $r = a \div (\cot \frac{1}{2} B + \cot \frac{1}{2} C).$
5. $r_1 = \Delta/(s-a).$
6. $r_1 = a \div (\tan \frac{1}{2} B + \tan \frac{1}{2} C).$

Using the above formulae, prove the following relations :

7. In a right-angled triangle $R + r = \frac{1}{2}(a + b)$.

8. $1/r_1 + 1/r_2 + 1/r_3 = 1/r$. 9. $1/r_2 + 1/r_3 = 2 \div b \sin c$.

10. $r r_1 r_2 r_3 = \Delta^2$. 11. $r r_1 = (s - b)(s - c)$.

12. $(abc \div \sin A \sin B \sin C)^{\frac{1}{3}}$. 13. $2Rr = abc \div (a + b + c)$.

14. $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$.

15. $\tan \frac{1}{2} A = \sqrt{(s - b)(s - c) \div s(s - a)}$.

16. $s^2 = \Delta \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C$.

17. If ABC is a triangle such that $2b = a + c$, and p is the length of the perpendicular from B upon AC , show that $\tan \frac{1}{2} A$ and $\tan \frac{1}{2} C$ are equal to the roots of the equation

$$x^2 - (b/p)x + \frac{1}{3} = 0.$$

18. Show that the sum of the radii of the escribed circles of a triangle is equal to the radius of the inscribed circle together with four times the radius of the circumscribing circle.

19. Show that the area of the triangle formed by joining the centres of the escribed circles is

$$8R^2 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C.$$

20. The sides of a triangle are 3, 5, 6; find the radii of the inscribed and circumscribed circles.

21. In an isosceles triangle the base is 100 cm. and the perpendicular from the vertex is 70 cm.; find the radii of the inscribed and circumscribed circles.

22. A triangle is described with base $BC = 5$ inches and angle $A = 70^\circ$. What is the radius of the circumcircle? Find the distance of the centre of the circumcircle from BC .

23. Find the radius of the circumcircle of the triangle ABC being given that $BC = 7$, $CA = 6$, and $C = 60^\circ$.

24. If $a = 32$, $b = 16$, $C = 42^\circ$, find R and r .

25. The area of a parallelogram having base 5.8 cm. and angle 123° is 37.7 sq. cm. Find the other sides and angles. Find the radii of the circles which pass through three of the corners of this parallelogram.

26. Two of the sides of a triangle are 7.5 cm. and 9.3 cm., the included angle is 37° . Find the radius of the circle which touches these sides produced and the third side.

Oral Revision Examples.

Complete the following identities and equations :

- | | |
|---|--|
| 1. $\sin(270 - A) =$ | 2. $\cos^2 \theta =$ |
| 3. $2 \tan A \cot A =$ | 4. If $\sin \theta = \frac{1}{2}$, $\theta =$ |
| 5. In any triangle $b^2 =$ | 6. In any triangle $R =$ |
| 7. $\sin 2A =$ | 8. $\tan 225^\circ =$ |
| 9. If $\cos \theta = \frac{2}{5}$, $\tan \theta =$ | 10. Δ in terms of the sides $=$ |
-
- | | |
|---|-----------------------------------|
| 11. $\tan^{-1} 1 =$ | 12. $\sec^2 A - 1 =$ |
| 13. length of arc $=$ radius $\times$ | 14. $\sin^2 B + \sin^2(90 - B) =$ |
| 15. Definition of tangent. | 16. In any triangle $\cos C =$ |
| 17. In any triangle $b \cos C + c \cos B =$ | |
| 18. In any triangle $bc \sin A =$ | 19. In any triangle $r =$ |
| 20. $\tan \frac{1}{2} \pi =$ | |
-
- | | |
|---|--|
| 21. Definition of sine. | 22. $\cos(360^\circ - B) =$ |
| 23. In any triangle $r_1 =$ | 24. In any triangle $\cos A =$ |
| 25. What formula connects a , b , and B ? | |
| 26. $\tan^{-1}(-\sqrt{3}) =$ | 27. If $\cos \theta = -\frac{1}{2}$, $\theta =$ |
| 28. $\tan^2 73\frac{1}{2}^\circ + 1 =$ | 29. $abc =$ |
| 30. $37^\circ = ?$ radians. | |
-
- | | |
|---|---|
| 31. $\cos^2(A - 45^\circ) + \sin^2(A - 45^\circ) =$ | |
| 32. $b \sin C =$ | |
| 33. Express R in terms of the sides. | |
| 34. If $\sin \theta = \sin \alpha$, then $\theta =$ | 35. $\cos \theta =$ (in terms of $\sin \frac{1}{2} \theta$). |
| 36. Area of triangle $=$ | 37. $a^2 + c^2 - 2ac \cos B =$ |
| 38. $\cos 1200^\circ =$ | |
| 39. Maximum value of $2 \sin \alpha \cos \alpha =$ | |
| 40. $\cos^4 \theta - \sin^4 \theta =$ (in its simplest form). | |
-
- | | |
|---------------------------------------|---------------------------------------|
| 41. $\Delta \div (s - a) =$ | 42. $\tan(180 - B) =$ |
| 43. $a \cos C + c \cos A =$ | 44. $bc \sin A =$ |
| 45. $\sin^2(A + B) + \cos^2(A + B) =$ | 46. If $\cos x = \cos A$, then $x =$ |
| 47. In any triangle $\cos A =$ | 48. $\tan 60^\circ =$ |
| 49. $\cos 2\theta =$ | 50. How many radians $= A^\circ$? |

Examples VI.

1. Prove that $(a \cos A - b \cos B) \div (a^2 - b^2) + \cos C/c = 0$.

2. Prove that $c^2 = (a+b)^2 \sin^2 \frac{1}{2} C + (a-b)^2 \cos^2 \frac{1}{2} C$.

3. In a triangle ABC the lines drawn from A and C , perpendicular to the opposite sides, intersect in O . If the angle A is acute, show that $OA = b \cos A / \sin B$.

Also draw a diagram in which A is an obtuse angle, and establish the corresponding expression for OA in that case.

4. Show that in any triangle the product of a side and the sines of the two adjacent angles is the same, whichever side be taken.

5. Find the area of a regular polygon of n sides circumscribed about a circle of radius r .

6. Regular polygons of 15 sides are inscribed in and circumscribed about a circle whose radius is one foot; show that the difference of their areas is nearly 20 square inches.

7. $ABCD$ are four points on a circle such that the angles BAC and BCA each equal θ . Show that $AD + CD = 2 BD \cos \theta$.

8. If $2 \cos B = \sin A / \sin C$, prove that the triangle is isosceles.

9. If $\tan A / \tan B = \sin^2 A / \sin^2 B$, show that the triangle is isosceles or right-angled.

10. Express the sides of a triangle in terms of the angles and the semi-perimeter.

11. In a triangle ABC perpendiculars AD and BE are let fall on the opposite sides; prove that the radius of the circle circumscribing the triangle CDE equals $R \cos C$.

12. If in a triangle the median bisecting the base AB is perpendicular to the side AC , prove that $2 \tan A + \tan C = 0$.

13. If p and q are the lengths of the perpendiculars from A, B on any arbitrary line drawn through the vertex C of a triangle, prove that

$$a^2 p^2 + b^2 q^2 - 2ab pq \cos C = a^2 b^2 \sin^2 C.$$

14. An isosceles triangle, vertical angle 35° , is inscribed in a circle whose radius is 1.65 inches. Find the lengths of the sides.

15. Show that in any triangle

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}.$$

16. If R is the radius of the circumcircle of any triangle and x, y, z are the lengths of the perpendiculars let fall from its centre on the sides, prove that

$$R^3 - (x^2 + y^2 + z^2)R - 2xyz = 0.$$

17. The rectangular co-ordinates of the angular points of a triangle are $(4, 5)$, $(6, 7)$, $(8, 6)$; determine the sum of the two smaller angles.

18. A rod AB , length $2a$, can turn about a hinge fixed to the wall at A ; it is supported by a string BC , length l , fastened to a point C on the wall at a height h above A .

(i) If BC is horizontal, what is the inclination of the rod to the vertical?

(ii) If BC is horizontal, what is the inclination to the vertical of the line joining the hinge to the middle point of the string?

(iii) If the string and rod are inclined at θ and ϕ to the vertical respectively, prove that (i) $2a \sin \phi = l \sin \theta$, (ii) $l \cos \theta - 2a \cos \phi = h$.

(iv) In the general case, what is the angle between the string and the rod? Give the answer in terms of h, a, θ or h, l, ϕ .

(v) In the general case, what is the inclination to the vertical of the line joining the hinge to the middle point of the string? Give the answer in terms of h, a, ϕ .

19. Three equal spheres of radius 7 centimetres are fixed in a horizontal plane so as to touch each other; a sphere of radius 6 cm. rests upon these three. Find the height of the centre of the fourth sphere above the horizontal plane, and the inclination to the vertical of the line joining the fourth centre to one of the lower centres.

20. Three equal rods of length 54 inches are fixed so as to form a tripod. If their feet are at the corners of an equilateral triangle, side 18 inches, find the inclination of each rod to the vertical.

21. In any triangle prove that the centroid trisects the line joining the circumcentre to the orthocentre.

22. Find the lengths of the sides of the pedal triangle of the triangle ABC . Find also the radii of the inscribed and circumscribed circles of that triangle.

(The pedal triangle is formed by joining the feet of the perpendiculars let fall from the vertices on the opposite sides.)

23. If $a = 5$ and $b = 4$, draw a graph to show the value of c as C varies from 0° to 180° . Hence find the value of c when $C = 40^\circ$.

CHAPTER VII

SOLUTION OF TRIANGLES

60. It is known from Geometry that, if three parts of a triangle are given, the remaining parts can in some cases be found; and that, in other cases, relations between the missing parts may be found even though their exact values cannot be determined. When actual numbers are given, results can be obtained to a greater degree of accuracy by Trigonometrical methods than by drawing to scale. In all cases a formula is sought which shall contain the three given letters and one unknown letter.

61. Case I. Three angles given.

The angle formula shows that $A + B + C$ must be 180° . No formula contains the three angles and one side only; but from the sine formula, viz. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$, we can find the ratios of the sides.

62. Case II. Two angles and one side given.

The third angle can be found immediately since

$$A + B + C = 180^\circ.$$

Suppose a is the given side; and it is required to find b . The formula must contain a , b , and two of the angles; hence we use

$$\frac{b}{\sin B} = \frac{a}{\sin A}.$$

This is adapted for the use of logarithms as it involves no addition or subtraction. If the tables in use give the logarithms of the cosecant, it may be advisable to use the following logarithmic form $\log b = \log a + \log \sin B + \log \operatorname{cosec} A$.

63. Case III. One angle and the two sides containing the angle are given.

Suppose a , b , C are the given parts. Then the cosine formula $c^2 = a^2 + b^2 - 2ab \cos C$ enables us to determine c . When c is

determined, the remaining angles can be found by the sine formula.

This method is of practical use only when the numbers involved are small; the cosine formula is not adapted for the use of logarithms. It is usual, therefore, to use the formula proved in

$$\S 53,* \text{ viz. } \tan \frac{1}{2}(A-B) = \frac{a-b}{a+b} \cot \frac{1}{2}C.$$

This determines $\frac{1}{2}(A-B)$; also $\frac{1}{2}(A+B)$ equals the complement of $\frac{1}{2}C$; hence A and B are found by adding and subtracting.

The value of c is then calculated by the sine formula.

64. Case IV. One angle and the two sides not containing the angle are given.

Suppose a, b, A are given. Then we can determine c from the formula

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This is a quadratic equation to determine c , and it is seen that there is the possibility of two distinct values for c . This is also seen from the geometrical construction. On this account this case is usually known as the **Ambiguous Case**.

If there are two values of c , there will be two values for B and for C . This is seen independently if the sine formula is used (as it usually is, on account of its adaptability for logarithms):

$$\frac{\sin B}{b} = \frac{\sin A}{a}.$$

Suppose that this leads to the result

$$\sin B = \sin x.$$

Then

$$B = x \text{ or } 180 - x.$$

This shows that, if there are two solutions, those two solutions are supplementary. Hence one of the solutions will be obtuse. Preliminary geometrical considerations often show that there can be only one solution.

(i) If the given angle A is not acute, then B must be acute and the obtuse-angled solution must be rejected.

(ii) If $a > b$ or $= b$, then $A > B$ or $= B$; consequently B cannot be obtuse.

Exercises. When a, b, A are given, show (i) from the geo-

* Another proof is given on p. 167.

metrical solution, (ii) from the cosine formula, (iii) from the sine formula, that

- (a) there is no solution, if $a < b \sin A$;
- (b) there is one solution only, if $a = b \sin A$;
- (c) there are two solutions, if $a > b \sin A$ but $< b$;
- (d) there is one solution only, if $a > b$.

Point out the difference in nature of the one solution in (b) and (d).

65. Case V. Three sides given.

Here again the cosine formula may be used, if the numbers involved are not inconveniently large. For logarithmic calculation the formula for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, or $\tan \frac{1}{2} A$ is used. These half-angle formulae are derived from the cosine formula.

$$2 \sin^2 \frac{1}{2} A = 1 - \cos A \quad (\S 51)$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b-c)^2}{2bc};$$

$$\therefore \sin^2 \frac{1}{2} A = \frac{(a-b+c)(a+b-c)}{4bc};$$

$$\therefore \sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Similarly $\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}.$

Divide $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$

Of these three formulae it is best to use the tangent formula; for the logarithms used in finding $\tan \frac{1}{2} A$ are the same as those required for finding $\tan \frac{1}{2} B$ or $\tan \frac{1}{2} C$. If only one angle has to be found, it is indifferent which formula is used.

There is a simple geometrical proof for $\tan \frac{1}{2} A$.

* In old books on Trigonometry the 'haversine' was used for solving triangles, and the values of log haversine were tabulated in mathematical tables. The haversine equals half the versed sine; hence $\text{haversin } A = \frac{1}{2} \text{versin } A = (1 - \cos A) \div 2 = \sin^2 \frac{1}{2} A$. The formula for solution of the triangle then becomes

$$\text{haversin } A = (s-b)(s-c) \div bc.$$

In Fig. XXXVIII I is the centre of the inscribed circle, Q is the point of contact of the circle with AC .

Then

$$\begin{aligned}\tan \frac{A}{2} &= \frac{IQ}{AQ} \\ &= \frac{r}{s-a} \\ &= \frac{\Delta}{s(s-a)} \\ &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.\end{aligned}\quad (\S 59)$$

Examples VII a. (See p. 81 for arrangement of work.)

In the following triangles when

Case I.

1. $A = 79^\circ 20'$, $B = 64^\circ 10'$, find the ratios of the sides.

Case II.

2. $A = 58^\circ 12'$, $B = 64^\circ 33'$, $a = 385$, find b .
3. $A = 38^\circ 24'$, $C = 95^\circ 5'$, $c = 7.832$, find a and b .
4. $B = 63^\circ 55'$, $C = 48^\circ 27'$, $c = 5.75$, find b .

Case III.

5. $a = 409$, $b = 381$, $C = 58^\circ 12'$, find A and B .
6. $B = 23^\circ 46'$, $c = 9.72$, $a = 8.88$, find A and C .
7. $a = .532$, $c = .259$, $B = 39^\circ 33'$, find A and C .
8. $A = 73^\circ 15'$, $b = 7315$, $c = 8013$, find B and C .

Case IV.

9. $A = 38^\circ 14'$, $a = .33$, $b = .44$, find C .
10. $a = 409$, $b = 385$, $A = 64^\circ 32'$, find B and C .
11. $b = 6.901$, $c = 5.749$, $C = 48^\circ 27'$, find B .
12. $A = 73^\circ 15'$, $a = 7315$, $c = 8013$, find B and C .

Case V.

13. $a = 17$, $b = 13$, $c = 12$, find the least angle.
14. $a = 793$, $b = 937$, $c = 379$, find all the angles.
15. $s = 1410$, $a = 1437$, $b = 811$, find all the angles.
16. $s = 1437$, $a = 1410$, $b = 811$, find all the angles.

17. $a = 13$, $b = 7$, $C = 60^\circ$, find A and B .
18. $a = 32$, $b = 40$, $c = 66$, find C .
19. $a = 250$, $b = 240$, $A = 72^\circ 4'$, find B and C .
20. $a = 2b$, $C = 120^\circ$, find A , B and the ratio of c to a .
21. $a = 36$, $b = 63$, $c = 81$, find the smallest angle.
22. $b = 5$, $c = 3$, $A = 42^\circ$, find B and C .

Oral Examples.

State the formula to be used in the following cases:

1. Given a , b , C , find c .
2. Given a , b , C , find A and B .
3. Given b , c , C , find B .
4. Given b , c , C , find a .
5. Given c , a , C , find A .
6. Given c , a , C , find B .
7. Given c , A , B , find C .
8. Given c , a , A , find b .
9. Given a , b , B , find C .
10. Given a , B , A , find c .
11. Given a , b , c , find C .
12. Given A , B , C , find a .
13. Given A , C , b , find a .
14. Given a , b , B , find A .
15. Given a , c , B , find C .
16. Given c , A , B , find b .
17. Given a , b , c , find B .
18. Given b , c , A , find B .
19. What is the ambiguous case?
20. When a , c , A are given, what are the conditions that there should be no ambiguity?

Examples VII b.

Solve the following triangles:

1. $a = 5$, $b = 7$, $C = 30^\circ$.
2. $b = 4$, $c = 3$, $C = 60^\circ$.
3. $a = 65$, $b = 63$, $c = 16$.
4. $b = 8$, $c = 9$, $C = 45^\circ$.
5. $a = 7$, $B = 120^\circ$, $A = 45^\circ$.
6. $a = 6$, $b = 7$, $c = 5$.
7. $b = 926.7$, $A = 48^\circ 24'$, $B = 31^\circ 13'$.
8. $a = 407.4$, $c = 115.9$, $A = 127^\circ 45'$.
9. $a = 1263$, $b = 1359$, $c = 1468$.
10. $a = 53.94$, $b = 156.5$, $C = 15^\circ 13'$.
11. $b = 457.2$, $c = 342.6$, $A = 73^\circ 45'$.
12. $a = 246.7$, $b = 342.5$, $B = 32^\circ 17'$.
13. $c = 79.48$, $A = 54^\circ 16'$, $B = 85^\circ 6'$.
14. $a = 7.956$, $b = 10.35$, $c = 9.412$.
15. $b = 9463$, $c = 7590$, $C = 43^\circ 47'$.
16. $a = 739$, $c = 937$, $B = 146^\circ 12'$.

$$\begin{array}{r}
 54^\circ 16' \\
 95^\circ 6' \\
 \hline
 149^\circ 22'
 \end{array}$$

Imposs

$$\begin{array}{r}
 100.05 \\
 92.19 \\
 \hline
 7.86
 \end{array}$$

17. $c = 79.5$, $A = 35^\circ 14'$, $C = 117^\circ 35'$.
 18. $A = 89^\circ$, $B = 18^\circ 47'$, $C = 72^\circ 13'$.
 19. $a = 87.6$, $b = 57.4$, $c = 46.8$. 20. $a = 79$, $c = 97$, $\Delta = 2437$.
 21. $A = 79^\circ$, $C = 97^\circ$, $R = 17.2$. 22. $b = 73.6$, $R = 57$, $a = 48.9$.
 23. $a^2 + b^2 = 841$, $\sin C = 1$, $\tan B = \frac{29}{21}$.
 24. $A = 42^\circ 35'$, $a = 83$, $b = 74$.
 25. $a = 2.740$, $b = .7401$, $C = 59^\circ 27'$.

Heights and Distances.

66. First a figure must be drawn, not necessarily to scale ; the known lengths and angles should be indicated in the figure. It may be necessary to solve, or partly solve, more than one triangle before the required measurement is found. The scheme for working should be carefully thought out before the work is actually begun.

Example i. *Wishing to find the height of a house standing on the summit of a hill of uniform slope, I descended the hill for 40 feet, and then found the height subtended an angle of $34^\circ 18'$. On descending a further distance of 60 feet, I found the subtended angle to be $19^\circ 15'$. Find the height of the house.*

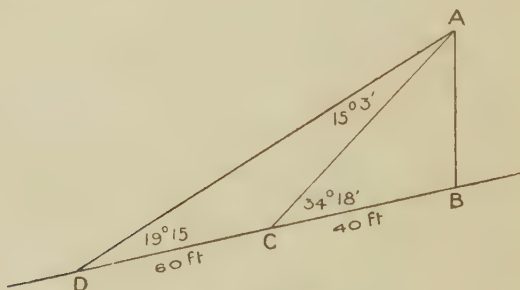


Fig. XXXIX.

Scheme.—In triangle ADC we know one side CD and all the angles ; so AC can be found. Then in the triangle ACB two sides AC , CB are known, and the included angle, hence AB can be found.

From triangle ACD ,

$$\frac{AC}{\sin ADC} = \frac{CD}{\sin DAC},$$

$$\text{i.e. } AC = 60 \sin 19^\circ 15' \operatorname{cosec} 15^\circ 3' \\ = 76.182.$$

From triangle ABC ,

$$\tan \frac{1}{2}(B-A) = \frac{b-a}{b+a} \cot \frac{1}{2}C \\ = \frac{36.182}{116.182} \cot 17^\circ 9';$$

$$\therefore \frac{1}{2}(B-A) = 45^\circ 16',$$

$$\frac{1}{2}(B+A) = 72^\circ 51'.$$

$$\therefore A = 27^\circ 35'.$$

$$\text{Again, } \frac{AB}{CB} = \frac{\sin ACB}{\sin CAB},$$

$$\text{i.e. } AB = \frac{40 \sin 34^\circ 18'}{\sin 27^\circ 35'} \\ = 48.792.$$

Height of house = 48.8 feet.

Logarithms.

$$1.77815$$

$$1.51811$$

$$.58559$$

$$1.88185$$

$$1.55849$$

$$+ .51061$$

$$2.06910$$

$$- 2.06514$$

$$.00396$$

$$1.60206$$

$$+ 1.75091$$

$$1.35397$$

$$- 1.66562$$

$$1.68835$$

Example ii. Wanting to know the height of a castle on a rock, I measured a base line of 100 yards, and at one extremity found the angle of elevation of the castle's top to be $45^\circ 15'$, and the angle subtended by the castle's height to be $34^\circ 30'$; also the angle subtended by the top of the castle and the other extremity of the base line was $73^\circ 14'$. At the other extremity the angle between the first extremity and the top of the castle was $73^\circ 18'$. Find the height of the castle.

This requires a rough perspective figure of the whole, and subsidiary plane figures.

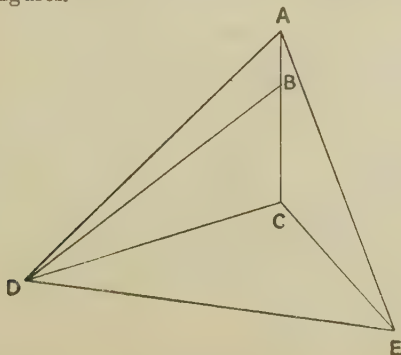


Fig. XL.

AB represents the castle.

C is the point in the same vertical as AB , and in the same horizontal plane as DE , the base line.

The following magnitudes are known :

$DE = 100$ yards.

ACD and ACE are each right angles.

ADC , ADB are known, therefore BDC is known.

ADE , AED are known.

Scheme. In triangle ADE , DE and the adjacent angles are known ; hence AD can be found. AB can now be found from triangle ABD .



Fig. XLI.

From triangle ADE ,

$$\frac{AD}{\sin 73^\circ 18'} = \frac{DE}{\sin 33^\circ 28'}$$

i.e. $AD = \frac{100 \sin 73^\circ 18'}{\sin 33^\circ 28'}$

$$\begin{array}{r} 2^\circ \\ + 1.9813 \\ - 1.7415 \\ \hline \log AD = 2.2398 \end{array}$$

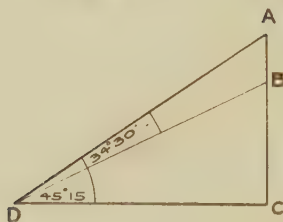


Fig. XLII.

From triangle ABD ,

$$AB = \frac{AD \sin 34^\circ 30'}{\sin 79^\circ 15'}$$

Height of castle = 100 yards.

$$\begin{array}{r} 2.2398 \\ + 1.7531 \\ \hline 1.9929 \\ - 1.9923 \\ \hline 2.0006 \end{array}$$

Example iii. *From the top of the Peak of Teneriffe the dip of the horizon is found to be $1^{\circ} 58'$. If the radius of the earth be 4000 miles, what is the height of the mountain?*

In Fig. XLIII C is the centre of the earth, AB is Teneriffe; BH is the tangent drawn from B to the earth's surface, so that H is the farthest point seen from B ; in other words, H is on the horizon. The angle between BH and BD (the perpendicular to the vertical) is called the dip of the horizon.

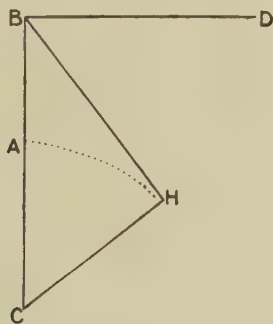


Fig. XLIII.

From triangle BCH ,

$$\frac{BC}{CH} = \sec BCH,$$

angle $BCH = \text{complement of } CBH = HBD;$

$$\begin{aligned} \therefore BC &= 4000 \sec 1^{\circ} 58' \\ &= 4000 \times 1.00059 \\ &= 4002.36 \text{ miles.} \end{aligned}$$

$\therefore$ Height of mountain is 2.36 miles.

Note. Fig. XLIII is drawn much out of scale; for small heights BH and BD are practically identical. Even for mountains the dip is very small, as in this example; in fact, so small that we may use the approximation $\sin \text{ of dip} = \tan \text{ of dip} = \text{circular measure of dip}$.

If E be the other extremity of the diameter through B , we have, from § 52,

$$BA \cdot BE = BH^2,$$

$$\text{i.e. } h(2r+h) = d^2,$$

where r is radius of earth, h is height of place of observation, d is the distance of the horizon.

$$\text{Hence } d = \sqrt{2rh + h^2};$$

$$\therefore d = \sqrt{2rh} \left(1 + \frac{h}{2r} \right)^{\frac{1}{2}}$$

$$= \sqrt{2rh} \left(1 + \frac{h}{4r} - \frac{h^2}{32r^2} \dots \right) \text{ by the Binomial Theorem.}$$

So far the work is accurate; usually h/r is so small that it may be neglected. Hence for ordinary heights

$$\text{Distance of horizon} = \sqrt{2rh}.$$

Exercise. (i) In the formula just obtained r , h , and the distance are all expressed in the same units. By taking $r = 3960$ miles, prove that

Distance of horizon in miles

$$= \sqrt{\frac{3}{2}} \times \text{height of place of observation in feet.}$$

(ii) Show also that

$$\text{Dip in minutes} = .9784 \sqrt{\text{height in feet.}}$$

Examples VII.

1. Standing at a horizontal distance 100 yards from the foot of a monument, a man observes the elevation of its top to be $25^\circ 35'$. Assuming the man's eye to be 5 feet from the ground, find the elevation of the top when the man stands 50 yards from the foot.

2. OA and OB are two straight roads intersecting at O and making with each other an angle of $35^\circ 12'$. A is a house 1572 yards from O , and B is a house 1129 yards from O . Find the direct distance between A and B .

3. A man observes the angles subtended by the base of a round tower at three points A , B , and C , in the same horizontal straight line with the centre of the circular base, to be 2α , 2β , 2γ respectively. Find the ratio of AB to BC , and find the diameter of the tower in terms of AC .

4. A man observes that the elevation of the top of a tower is $37^{\circ} 40'$, and that the elevation of the top of a flagstaff on the tower is $43^{\circ} 59'$; show that the height of the flagstaff is one-fourth of the height of the tower very nearly.

5. Having given that the least side of a triangle is 17.3 inches, and that two of the angles are $63^{\circ} 20'$ and $72^{\circ} 40'$, find the greatest side.

6. If two sides of a triangle are 7235 feet and 4635 feet respectively, and if the included angle is $78^{\circ} 26'$, find the remaining angles of the triangle.

7. The base of a triangle being 7 feet, and the base angles $129^{\circ} 23'$ and $38^{\circ} 36'$, find the length of the shorter side.

8. Explain the ambiguous case of the solution of triangles. When a, b, A are given and the question is asked whether, from these data, two triangles, one triangle, or no triangle can be constructed, show that the question can be answered from a consideration of the roots of the equation

$$x^2 - 2bx \cos A + b^2 = a^2.$$

9. From each of two ships, a mile apart, the angle is observed which is subtended by the other ship and a beacon on shore; these angles are found to be $52^{\circ} 25'$ and $75^{\circ} 10'$ respectively. Find the distances of the beacon from each of the ships.

10. A ship sailing due north observes two lighthouses bearing respectively NE. and NNE. After the ship has sailed 20 miles the lighthouses are seen to be in a line due east. Find the distance in miles between the lighthouses.

11. The angles A, B, C of a triangle ABC are $40^{\circ}, 60^{\circ},$ and 80° respectively, and CD is drawn from C to the base bisecting the angle ACB ; if AB equals 100 inches, find the length of CD .

12. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at 45° to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the angles of inclination are 15° and 75° respectively. Find the distance between the boats and the perpendicular distance of each from the sea-wall.

13. From a house on one side of a street observations are made of the angle subtended by the height of the opposite house, first

from the level of the street, in which case the angle is $\tan^{-1}(3)$, and afterwards from two windows, one above the other, from each of which the angle is found to be $\tan^{-1}(-3)$. The height of the opposite house being 60 feet, find the height of each of the two windows above the street.

14. A segment of a circle stands on a chord AB 10 cm. long and contains an angle of 40° . A point C travels along the arc; for what value of the angle ABC is the chord CA three times the chord CB ? Verify by drawing a graph showing the chord CA as a function of the chord CB .

15. If the sides of a triangle are 1011 and 525 feet, and the difference of the angles opposite to them is 24° , find (correct to the nearest degree) the smallest angle of the triangle.

16. A ladder is placed against the wall of a room and is inclined at an angle α to the floor. If the foot of the ladder slips outwards from the wall a distance of a feet, and the inclination of the ladder to the floor is then β , show that the distance which the top of the ladder will slide down the wall is $a \cot \frac{1}{2}(\alpha + \beta)$.

17. A man travelling due west along a straight road observes that when he is due south of a certain windmill the straight line drawn to a distant tower makes an angle of 30° with the direction of the road. A mile further on the bearings of the windmill and tower are NE. and NW. respectively. Find the distances of the tower from the windmill, and from the nearest point of the road.

18. A statue 10 feet high, standing on a column 100 feet high, subtends at the eye of an observer in the horizontal plane from which the column springs the same angle as a man 6 feet high standing at the foot of the column; find the distance of the observer from the column.

19. It is found that two points, each 10 feet from the earth's surface, cease to be visible from each other over a level plain at a distance of 8 miles; find the earth's diameter.

20. A plane, inclined at 33° to the horizontal, meets a horizontal plane in the line BC . From B a line BD is drawn on the inclined plane making an angle 27° with the horizontal plane. If BD is 18 inches long, find the height of D above the horizontal plane, and its distance from BC . Also find the angle BD makes with BC .

21. A lighthouse was observed from a ship to be $N. 23^{\circ} E.$; after the ship had sailed due south for 3 miles, the same lighthouse bore $N. 12^{\circ} E.$ Find the distance of the lighthouse from the latter position of the ship.

22. Two streets meet at an acute angle; the one lies $N. 51^{\circ} W.$, and the other $S. 48^{\circ} W.$ The distance from the corner to a chemist's door in the first street is 315 yards; and the distance from the corner to a doctor's door in the other street is 406 yards. Find the length of a telephone wire going direct from the doctor's house to the chemist's.

23. From a vessel at anchor two rocks are observed to the westward, the one (A) bearing $WNW.$, and the other (B) $W. by S.$ from the vessel. From the chart it is found that A bears $NNE.$ from B and is distant 645 yards from it. What are the distances of the rocks from the vessel?

24. Three objects A , B , and C forming a triangle are visible from a station D at which the sides subtend equal angles. Find AD , it being known that

$$AB = 12 \text{ miles, } AC = 6 \text{ miles, } CAB = 46^{\circ} 34'.$$

25. A tower on the bank of a river, whose breadth is 100 feet, subtends angles $22\frac{1}{2}^{\circ}$ and $67\frac{1}{2}^{\circ}$ at two points A and B on the opposite bank of the river, whose distance apart is 600 feet, on a level with the base of the tower. Find the height of the tower.

*26. A , B , C are three given stations, so that the triangle ABC is completely known. Show how to determine, by means of angles measured at a fourth station P , the distances PA , PB , PC , the four stations being all in one plane, the case for consideration being that in which P is within the angle A , and the points P and A on opposite sides of BC .

If ABC is equilateral, and the angle BPC equals 60° , show that

$$2 \cos (60^{\circ} + BAP) + \cos (ABP - BPA) = 0.$$

27. A tower stands on the edge of a circular lake $ABCD$. The foot of the tower is at D , and the angles of elevation of the top of the tower from A , B , C , are α , β , γ respectively. If the angles BCA , BAC be each equal to θ , show that

$$\cotan \alpha + \cotan \gamma = 2 \cotan \beta \cos \theta.$$

* This example is best solved by using the formulae of §§ 83 and 84.

28. A mountain is observed from a place A to have elevation $15^{\circ} 17'$ and to bear $N. 24^{\circ} 29' W.$ From another place B which is 2347 yards north of A its bearing is $N. 37^{\circ} 2' W.$ Deduce the elevation from B .

29. The extremity of the shadow of a flagstaff 6 feet high, standing on the top of a regular pyramid on a square base, just reaches a side of the base and is distant 56 feet and 8 feet from the extremities of that side. If the height of the pyramid be 34 feet, find the sun's altitude.

30. A man observes that when he has walked c feet up an inclined plane the angular depression of an object in the horizontal plane through the foot of the slope is α ; and that, when he has walked a further distance of c feet, the angular depression of the object is β . Show that the inclination of the slope to the horizon is $\cot^{-1}(2 \cot \beta - \cot \alpha)$; and determine the distance of the object observed from the foot of the slope.

31. A straight flagstaff, leaning due east, is found to subtend an angle α at a point in the plain upon which it stands, a yards west of the base. At a point b yards east of the base, the flagstaff subtends an angle β . Find at what angle it leans.

32. Four rods are loosely jointed at their extremities to form a parallelogram with sides 4 and 5 inches long. Two of the opposite corners are connected by an elastic string of length 7 inches. Find the angle between the string and the shorter side.

If the length of the other diagonal be diminished by 1 inch, what does the angle become?

*33. Three posts on the border of a lake are at known distances from each other, namely 63 yards, 44 yards, and 76 yards. At a boat on the lake it is found that the two posts, whose distance is 63 yards, subtend an angle $89^{\circ} 15'$, and the two posts, whose distance is 76 yards, subtend an angle $130^{\circ} 45'$. Find the distances of the boat from the three posts.

34. A base line AB is drawn 2 chains in length on a plane in the same horizontal plane as C the foot of a tree. The angles ABC , BAC are found to be $79^{\circ} 56'$ and $78^{\circ} 18'$ respectively; the angle of elevation of the top of the tree is found to be $19^{\circ} 46'$ at A . Find the height of the tree to the nearest foot.

* This example is best solved by using the formulae of §§ 83 and 84.

35. A base line AB , 2527 links long, is measured on the sea-shore along the high water mark. C is a point where a distant rock meets the sea; the angles BAC , ABC are found to be $89^\circ 15'$, $86^\circ 21'$ respectively. The angle of elevation of the highest point of the rock, which is vertically above C , as observed at A , is $1^\circ 48'$. Neglecting the curvature of the earth, find the height of the rock and its distance from A .

36. A hill slopes upwards towards the North at an inclination 14° to the horizontal. The sun is 15° W. of S., at an altitude of 47° ; find the length of the shadow cast on the hill by a vertical post 39 feet high.

37. If, in the previous question, the post is perpendicular to the surface of the hill, what is the length of the shadow?

Revision Examples B.

1. Define the tangent of any angle, and prove from the definition that (i) $\tan(90 + A) = -\cot A$; (ii) $\tan(180 - A) = -\tan A$. Express the other trigonometrical ratios in terms of the tangent.

2. Show by substitution that

$$\sin 45^\circ + \sin 30^\circ > \sin 60^\circ,$$

and

$$\cos 30^\circ - \cos 45^\circ < \cos 60^\circ.$$

3. Find the value of $\sin 45^\circ$ without using tables.

Solve the equation $4 \sin \theta \cos \theta + 1 = 2(\sin \theta + \cos \theta)$.

Give the general solutions.

4. A man walks directly across the deck of a ship, which is sailing due North at 4 miles an hour, in 12 seconds, and finds that he has moved in a direction 30° East of North. How wide is the deck?

5. Show that in any triangle ABC ,

$$(i) \sin A/a = \sin B/b = \sin C/c;$$

$$(ii) \sin C(a \cos B - b \cos A) = (a + b)(\sin A - \sin B).$$

6. Prove geometrically that

$$\cos 2A = 1 - 2\sin^2 A.$$

Hence find the value of $\sin 15^\circ$.

7. The angle of elevation of the top of a spire seen from A is 30° , and it is found that at a point B , $115\frac{1}{2}$ feet nearer the foot of the spire, it is 60° . Find the height of the spire to the nearest foot.

8. Plot a curve giving the sum of $4 \sin \theta$ and $3 \sin 2\theta$ from $\theta = 0^\circ$ to $\theta = 180^\circ$; and read off the angles at which the greatest and least values respectively of this sum occur.

Estimate the slope of the curve when $\theta = 90^\circ$ and when $\theta = 135^\circ$.

9. Define a radian. Express in degrees and minutes an angle of 1.36 radians.

Find the number of radians in the angle of a regular decagon.

10. Prove

$$(i) \sin^2 A + \cos^2 A = 1;$$

$$(ii) \tan A \div (1 - \cot A) + \cot A \div (1 - \tan A) = \sec A \operatorname{cosec} A + 1.$$

11. Draw the sine and cosine graphs, in the same figure, from $\theta = 10^\circ$ to $\theta = 20^\circ$.

From the graph find the angle which satisfies

$$\sin \theta + \cos \theta = 1.2.$$

12. Find an expression which will include all angles having a given tangent. Write down the values of $\tan 225^\circ$, $\tan 780^\circ$, $\cot 1035^\circ$, $\cot 210^\circ$.

Construct an angle, having given the cotangent.

13. Find $a/\cos A + b/\cos B + c/\cos C$ in a form adapted to logarithmic calculation.

14. In any triangle prove that (i) $a = b \cos C + c \cos B$; (ii) $a(b \cos C - c \cos B) = b^2 - c^2$; (iii) $r \cos \frac{1}{2} A = a \sin \frac{1}{2} B \sin \frac{1}{2} C$.

15. If the sides of a parallelogram are a , b , and the angle between them ω , prove that the product of the diagonals is

$$\sqrt{a^4 - 2a^2b^2 \cos 2\omega + b^4}.$$

16. A vessel is steaming towards the East at 10 miles an hour. The bearing of a lighthouse as seen from the vessel is $42^\circ 24'$ North of East at noon, and $25^\circ 12'$ East of North 25 minutes later. Find how far the vessel was from the lighthouse at noon, and find also at what time the bearing of the lighthouse will be due North.

17. Assuming that a circle may be treated as a regular polygon with an infinite number of sides, show that the ratio of the circumference of a circle to its diameter is constant.

What is the circular measure of the least angle whose sine is $\frac{1}{2}$, and what is the measure in degrees, &c., of the angle whose circular measure is .15708?

18. Prove by a geometrical construction that

$$\cos 2A = \cos^2 A - \sin^2 A.$$

Solve the equation $\cos 2A = (\cos A + \sin A)^2$.

19. For what data will the solution of a triangle become ambiguous? Explain this.

Given $B = 30^\circ$, $c = 150$, $b = 50\sqrt{3}$, show that of the two triangles that satisfy the data one will be isosceles and the other right-angled. Find the third side in the greater of these triangles.

Would the solution be ambiguous if $B = 30^\circ$, $c = 150$, $b = 75$?

20. AB is a horizontal line whose length is 400 yards; from a point in the line between A and B a balloon ascends vertically,

and after a certain time its altitude is taken simultaneously from A and B ; at A it is observed to be $64^\circ 15'$; at B $48^\circ 20'$; find the height of the balloon.

21. Find the radius of the circle circumscribing a triangle, in terms of its sides. If $c^2 = a^2 + b^2$, show that this radius equals $\frac{1}{2}c$.

22. Define the trigonometrical ratios of A involved in the equation $\cot A + \tan A = \sec A \operatorname{cosec} A$; and establish its truth by a geometrical construction.

23. Prove that

$$\begin{aligned}\cos^{-1} \sqrt{(a-x) \div (a-b)} &= \sin^{-1} \sqrt{(x-b) \div (a-b)} \\ &= \cot^{-1} \sqrt{(a-x) \div (x-b)}.\end{aligned}$$

24. Prove that $\sin \theta = \tan \theta \div \sqrt{1 + \tan^2 \theta}$.

Having given $\tan \theta = \frac{3}{4}$, find $\sin \theta$, $\cos \theta$, and $\operatorname{versin} \theta$.

25. If θ is an acute angle whose sine is $\frac{8}{17}$, calculate the value of $\tan \theta + \sec \theta$.

What would the value be if θ were obtuse?

26. What is the angle between the diagonal of a cube and one of the edges at its extremity?

27. Obtain an expression for all the angles which have a given tangent.

Find all the angles lying between -360° and $+860^\circ$ which satisfy the equation

$$\tan^2 x - \frac{2}{\sqrt{3}} \tan x - 1 = 0.$$

28. A circular wire of 3 inches radius is cut and then bent so as to lie along the circumference of a hoop whose radius is 4 feet. Find the angle which it subtends at the centre of the hoop.

29. A triangle ABC has angle $A = 34^\circ$, $a = 11.0$ cm., $c = 7.8$ cm. Calculate the perpendicular from B on b , and the remaining angles and side of the triangle.

30. In a triangle $a = 74$, $b = 37$, $c = 97$; find the value of (i) $a \cos B + b \cos A$, (ii) $a \sin B - b \sin A$.

31. If ABC be a triangle, and θ an angle such that

$$\sin \theta = 2\sqrt{ab} \cos \frac{1}{2}C \div (a+b),$$

find c in terms of a , b , and θ .

If $a = 11$, $b = 25$, and $C = 106^\circ 15\frac{1}{2}'$, find c .

32. Find the area of a regular quindecagon inscribed in a circle of one foot radius.

33. Find an expression for all angles having the same sine as the angle α .

Solve the equation $\sin(\alpha + x) + \sin(\beta + x) = 0$.

34. An angle α is determined by the equations $v^2 = 2gh$, $-b = tv \sin \alpha - \frac{1}{2}gt^2$, $tv \cos \alpha = a$. Show that

$$a^2 \tan^2 \alpha - 4ha \tan \alpha + a^2 - 4hb = 0.$$

35. Criticize the proposition that three measurements are sufficient and necessary to determine a triangle uniquely in shape and size.

36. A square house, measuring 30 feet each way, has a roof sloping up from all four walls at 35° to the horizontal. Find the area of the roof.

37. Draw up a table showing in three columns the values of $10 \sin \theta$, $10 \cos \theta$, and $8 \sin \theta + 6 \cos \theta$ for each 30° from 0° to 360° . From the table draw, in the same figure, the graphs of $y = 10 \sin \theta$ and $y = 8 \sin \theta + 6 \cos \theta$; and from the curves determine approximately a value of θ for which $\tan \theta = 3$.

38. Taking the earth as a sphere of radius 4000 miles, find the distance London travels in an hour in consequence of the rotation of the earth. (Latitude of London $51^\circ 30' \text{ N.}$)

39. $ABCD$ is a quadrilateral in which AB and DC are parallel and 40 feet apart, and AB is 100 feet long. The angle DAB is $72^\circ 30'$, and the angle CBA is $38^\circ 15'$. Find the lengths of AD , DC , and CB , and the area of the quadrilateral.

40. State the local time at the following places when it is noon at Greenwich.

Cape Town $33^\circ 56' \text{ S.}$, $18^\circ 25' \text{ E.}$ Fiji $18^\circ 0' \text{ S.}$, $178^\circ 0' \text{ E.}$

Edinburgh $55^\circ 57' \text{ N.}$, $3^\circ 10' \text{ W.}$ Singapore $1^\circ 17' \text{ N.}$, $103^\circ 50' \text{ E.}$

41. Define the cosine and the tangent of an angle, and show how to express the tangent in terms of the cosine.

Having given that $\cos A = .8$, and that A is less than 90° , find the value of $\tan A$; and by means of the tables find the value of A , both from its cosine and from its tangent.

42. Prove that, in any triangle ABC , $\sin B : \sin C = b : c$. In the triangle ABC the angle CAB is 50° , the angle ABC is 65° , and the side BC is 4 inches long. Find the length of the side AB .

43. Show how to find the height of a tree by means of a chain for measuring lengths and of an instrument for measuring angles.

44. Find an expression for all the angles which have (i) a given tangent, (ii) a given sine.

45. Explain how it is that, $\tan \theta$ being given, $\tan 2\theta$ is known; but that, $\sin \theta$ being given, $\sin 2\theta$ may have either of two values.

46. Prove that the area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. Show also that the area is $\frac{1}{2}c^2 \div (\cot A + \cot B)$.

47. Find the radius of the circumscribing circle of the triangle for which $A = 66^\circ 30'$, $B = 11^\circ 30'$, $c = 200$ feet.

48. A ship is sailing due East at a uniform rate: a man on a lighthouse observes that it is due South at 1 p.m. and $16^\circ 30'$ East of South at 1.20 p.m. In what direction will he see it at 2 p.m.?

CHAPTER VIII

PROJECTION. VECTORS

67. If from the extremities of a line PQ , of definite length, perpendiculars PK , QL are let fall on a line AB , which may be produced if necessary, then KL is called the **Projection of PQ on the line AB .**

Projections are subject to the same convention of sign as are abscissae and ordinates. Thus, in the above figure, KL is positive, but LK is negative. It follows that the projection of PQ is not the same as the projection of QP , so that the order of the letters in naming a line is of great importance when we are

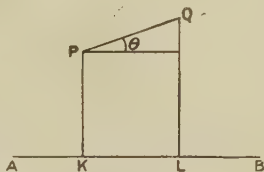


Fig. XLIV.

dealing with projection. When the direction of the line is to be taken into account as well as its length, it is called a **directed length**; and we shall, in future, use the symbol $(PQ)^*$ to denote the directed length of the line from P to Q . The number of units of length in that line we shall continue to denote by the symbol PQ .

Thus, in Fig. XLIV, the projection of (PQ) is (KL) ,
and the projection of (QP) is (LK) .

Note. When we speak of the sum of directed lengths in the same straight line, the algebraical sum is always meant. Geometrically this means that we require the directed length between the starting-point and final point, and not the length of the actual path traversed.

68. If the length of PQ is l , and if θ is the angle between PQ and the line AB , then

$$\text{projection of } (PQ) \text{ on } AB = l \cos \theta.$$

* This is usually written $\overrightarrow{PQ}$.

Some care is necessary in applying this formula; the safest plan is to keep l and θ both positive.

Consider, for instance, the projection of (QP) in Fig. XLV.

Imagine a line drawn from the initial point Q parallel to the line AB . Then it is seen that the angle between (QP) and AB is $\theta + \pi$, while the length QP is l .

Hence projection of (QP) on $AB = l \cos(\theta + \pi) = -l \cos \theta$.

Two other methods of treatment give the same result.

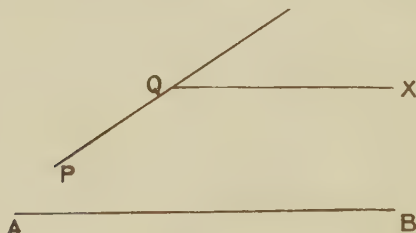


Fig. XLV.

In Fig. XLV the line QX is actually drawn parallel to AB ; but it is usually sufficient to imagine it. Then we may take the angle between (QP) and QX to be the negative angle XQP , i.e. $-(\pi - \theta)$; the length QP is positive so that

$$\text{projection of } QP = l \cos(-\pi - \theta) = -l \cos \theta.$$

Or we may regard θ as being the angle between (QP) and QX ; but this requires that the length of (QP) should be taken as $-l$, and so the projection of (QP) on $AB = -l \cos \theta$.

It will be found that, in all cases, $l \cos \theta$ gives both the magnitude and sign of the projection of (PQ) on AB . Similarly,

$$\begin{aligned} \text{the projection of } (PQ) \text{ on a line perpendicular to } AB \\ = l \sin \theta. \end{aligned}$$

69. Proposition A. *The sum of the projections on any*

line of two sides (AB) , (BC) of a triangle is equal to the projection of the third side (AC) .

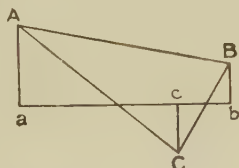
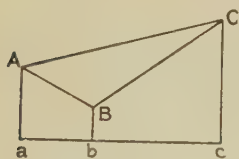


Fig. XLVI.

In either of the above figures (or in any other figure)
 projection of (AB) + projection of (BC) = $(ab) + (bc)$
 = (ac)
 = projection of (AC) .

Proposition B. *The sum of the projections on any line of the three sides (AB) , (BC) , (CA) of a triangle is zero.*

Sum of projections of (AB) , (BC) , (CA) = $(ab) + (bc) + (ca)$.

Hence on the line of projection we start at the point a and finish at the same point, so that the distance between the initial and final points is zero. That is, the sum of the projections is zero.

Proposition C. *In any closed figure $ABC \dots HK$, the sum of the projections of the sides (AB) , $(BC) \dots (HK)$ equals the projection of (AK) .*

Proposition D. *In any closed figure the sum of the projections of all the sides taken in order in the same direction is zero.*

Propositions C and D are proved exactly in the same way as Propositions A and B.

Example. *Prove that*

$$\cos A + \cos (120 + A) + \cos (120 - A) = 0.$$

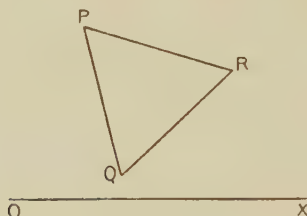


Fig. XLVII.

Draw an equilateral triangle PQR , side a units.

Draw a line OX inclined at an angle A to (QR)

Then (RP) is inclined at $A + 120$ degrees to OX ; and (PQ) is inclined at $A + 240$ degrees.

Project on OX ; then, by Proposition B,

$$a \cos A + a \cos (A + 120) + a \cos (A + 240) = 0;$$

but $\cos (A + 240) = \cos \{360 - (120 - A)\} = \cos (120 - A)$;

$$\therefore \cos A + \cos (120 + A) + \cos (120 - A) = 0.$$

Examples VIII a.

(These examples should be verified by drawing a figure to scale.)

1. Show that the projection of a line on a line parallel to itself is equal to the projected line, and that the projection of a line on a line perpendicular to itself is zero.

2. A line of length r , making an angle θ with OX is projected on OX and at right angles to OX ; calculate the lengths of the projections in the following cases:

- | | |
|--|--|
| (i) $r = 5$, $\theta = 60^\circ$; | (ii) $r = -5$, $\theta = 120^\circ$; |
| (iii) $r = 5$, $\theta = 248^\circ$; | (iv) $r = 5$, $\theta = 300^\circ$; |
| (v) $r = -5$, $\theta = 330^\circ$. | |

3. Two rods AB , BC , of lengths 5 feet and 10 feet respectively, are joined together at an angle of 135° . The rods are fixed in

a vertical plane so that CB is inclined at 60° to the horizontal, and the angle ABC is beneath the rods; by projecting horizontally and vertically, find the inclination of the line AC to the horizontal.

4. By projecting a diagonal and two sides of a square on a line making an angle A° with one of the sides, prove that

$$\cos(A + 45^\circ) = (\cos A - \sin A) \div \sqrt{2}.$$

Find a similar value for $\sin(A + 45^\circ)$.

5. PQR is a triangle right-angled at Q , having the angle at P equal to A° ; PQ is inclined to OX at an angle B° .

Prove by projection that

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

$$\text{and} \quad \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

70. If the projections of a line on two lines at right angles are given, the length and direction of the projected line can be found, but not its actual position.

Let r be the length of the line and θ the angle it makes with one of the lines of projection. Then $r \cos \theta$ and $r \sin \theta$ are known; suppose these values are x and y respectively, so that $r \cos \theta = x$ and $r \sin \theta = y$.

$$\text{Then } r = \sqrt{x^2 + y^2} \text{ and } \tan \theta = y/x.$$

The projected line has therefore a definite length and a definite direction; it is the simplest example of a group of quantities called **vector quantities** or **vectors**.

71. A quantity which possesses a direction as well as magnitude is called a **vector**. Velocities and forces are examples of such quantities. The magnitude and direction can be represented by the length and direction of a directed straight line; hence the properties of a directed straight line that depend only on its length and direction represent properties common to all vectors.

72. Vector addition or Composition of Vectors.

A displacement from A to B followed by a displacement from B to C produces the same result as a single displacement from A to C .



Fig. XLVIII.

Or we may regard the displacements as being simultaneous.

Suppose a point to start from A and move along AB , and while this point is moving, suppose the line AB to move parallel to itself, the point B moving to C while the point travels from A to B . The result of the two simultaneous displacements is that the point has travelled from A to C .

Hence the vector (AC) is called the resultant of the vectors (AB) and (BC) .

Finding one quantity equivalent to two or more of the same kind is equivalent to the process of addition in Arithmetic.

If we use the sign $+$ to denote this process, we have

$$(AC) = (AB) + (BC).$$

If P , Q , and R are the respective magnitudes of the vectors represented by (AB) , (BC) , and (AC) , and if θ is the angle between the directions of (AB) and (BC) (in Fig. XLVIII the angle ABC is the supplement of θ);

$$\text{then} \quad R^2 = P^2 + Q^2 + 2PQ \cos \theta.$$

Similarly, if a number of vectors are represented by the directed lengths (AB) , (BC) , $(CD) \dots (HK)$, then their resultant is represented by the directed length (AK) .

73. Resolution of vectors.

In Fig. XLVIII the vector (AC) may be replaced by the two vectors (AB) and (BC) . Viewed in this light they are called the **components** of the vector (AC) .

When we talk of the component of a vector in a given direction, and no mention is made of the direction of the other component, it is understood that the other component is at right angles to the first.

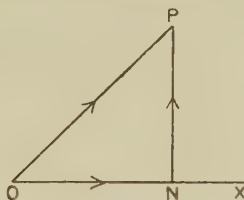


Fig. XLIX.

If (OP) in Fig. XLIX represents a vector of magnitude R inclined at an angle θ to OX , then its projection (ON) represents the component along OX , and the projection (NP) represents the component perpendicular to OX .

The vector is now said to be **resolved** along and perpendicular to OX .

Resolving along OX , we find that the component is $R \cos \theta$.

Resolving perpendicular to OX , we find that the component is $R \sin \theta$.

74. All the work of § 69 on projections can be applied to vectors and their components. For instance, Proposition C gives the following proposition:

The sum of the components of any number of vectors in a given direction is equal to the component of their resultant in that direction.

Examples VIII b.

[In the following examples the letters P, Q, R imply that the vectors are forces; the letters u, v, w imply that the vectors are velocities. When possible, figures should be drawn to scale to check the calculation.]

1. Find the resultant R in the following cases:

(i) $P = 17, Q = 13, \theta = 40^\circ$;

(ii) $P = 17, Q = 13, \theta = 140^\circ$;

(iii) $P = 114, Q = 75, \theta = 65^\circ$;

(iv) $P = 123, Q = 496, \theta = 117^\circ$.

2. Find P when Q the other vector, θ the angle between them, and R their resultant have the following values:

(i) $Q = 176, R = 249, \theta = 72^\circ$;

(ii) $Q = 73, R = 193, \theta = 110^\circ$;

(iii) $Q = 245, R = 92, \theta = 130^\circ$;

(iv) $Q = 36, R = 84, \theta = 20^\circ$.

3. Show that, if the resultant of three forces is zero, the sum of their components in any direction is zero.

4. Show that if three forces produce equilibrium (their resultant is, therefore, zero) they are parallel and proportional to the sides of a triangle.

5. A boat is being rowed due E. at a speed of 6 miles an hour; at the same time a current carries it due S. with a speed of 3 miles an hour; find the magnitude and direction of the actual velocity.

6. Find the resultant of velocities u and v inclined at an angle θ , when

(i) $u = 14, v = 16, \theta = 180^\circ$;

(ii) $u = 14, v = 16, \theta = 65^\circ$;

(iii) $u = 14, v = 16, \theta = 135^\circ$.

7. Vectors of magnitudes 7, 8, 9 respectively are parallel to three consecutive sides of a regular hexagon. Find the sum of their components (i) parallel to, (ii) perpendicular to, the middle one of these sides. Hence find the magnitude and direction of their resultant.

8. Find the magnitude and direction of the resultant of four forces of magnitudes 5, 10, 15, 20 respectively, which act along the sides of a square.

9. A stream flows at the rate of 2 miles an hour. In what direction must a man swim in order that he may actually go straight across the river, his rate of swimming being 3 miles an hour?

10. A rod 5 feet long is hung by a string, attached to its two ends, over a smooth peg; it rests, at an angle of 20° to the horizontal, so that the two portions of the string are each inclined 35° to the vertical. Find the length of the string.

Projection on a Plane.

75. If from every point in a line, straight or curved, a perpendicular be let fall on a plane, the locus of the feet of the perpendiculars is called the projection of the line on the plane.

If from every point in the boundary of a surface a perpendicular be let fall on a plane, the area bounded by the locus of the feet of the perpendiculars is called the projection of the surface on the plane.

76. The angle between a straight line and its projection on a plane is called the angle between the straight line and the plane. It follows that the projection on a plane of a straight line of length l , making an angle α with the plane, is $l \cos \alpha$.

Any two planes, not parallel, intersect in a straight line. If from any point P in this line two perpendiculars PA , PB are drawn to it, one in each plane, then the angle APB measures the angle between the planes.

77. If any plane surface, of area A , is projected on a plane making an angle α with its own plane; then the area of the projection is $A \cos \alpha$.

*Step I. Consider a rectangle $ABCD$, having the side AB parallel to the plane of projection, and the side BC making an angle α with that plane; then α is the angle between the plane of the rectangle and the plane of projection.

Then, in Fig. I, $abcd$ is the projection of $ABCD$.

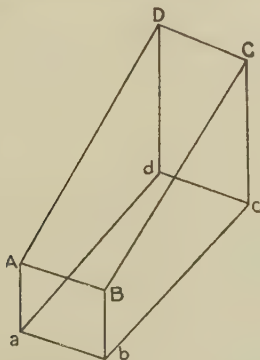


Fig. I.

* A slight knowledge of solid geometry is assumed in this proof.

Now Bb is perpendicular to the plane $abcd$, and therefore to the line ab ;

$\therefore Bb$ is perpendicular to AB ;

but BC is perpendicular to AB ;

$\therefore AB$ is perpendicular to plane $BCcb$;

$\therefore ab$ is perpendicular to plane $BCcb$;

$\therefore ab$ is at right angles to bc ,

i.e. $abcd$ is a rectangle.

Hence area of $abcd = ab \times bc$

$$= AB \times BC \cos \alpha$$

$$= \text{area of } ABCD \times \cos \alpha.$$

Step II. Consider a plane area with curved or rectilinear boundary. In the plane of the figure draw any line PQ parallel to the plane of projection. Then in the area we can inscribe a number of rectangles having the short sides parallel to PQ and the longer sides perpendicular to PQ .

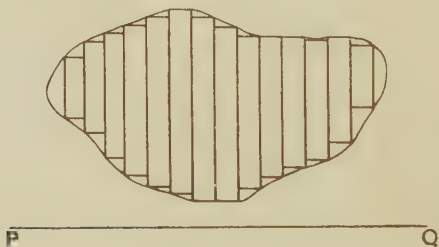


Fig. LI.

The sum of these rectangles is less than the original area, but may be made to differ from that area by as small a quantity as we please by making their width small enough; and then the sum of their projections will differ from the projection of the area by an even smaller quantity. Hence in the limit, when the width is indefinitely small, the sum of each set of rectangles will equal the area of the corresponding circumscribing figure.

But the sum of projections of rectangles = sum of rectangles $\times \cos \alpha$;

$\therefore$ the area of projected figure = area of original figure $\times \cos \alpha$.

Examples VIII c.

1. A pyramid $VABCD$ has a square base $ABCD$, side a , and the faces VAB , &c., are equilateral triangles. Find the length of the projection of VA on the base.

Verify that the sum of the areas of the projections of the four faces is equal to a^2 .

2. A square house, whose side is 28 feet long, has a roof sloping up from all four walls at 40° to the horizontal, find the area of the roof.

3. Find, by projection, the area of the curved surface of a right circular cone, having height h , and semi-vertical angle 2α .

4. From a cone 6 feet high a smaller cone 2 feet high is cut off. If the radius of the base of the small cone is 1.6 feet, find the area of the curved surface of the remainder of the large cone.

Verify your answer by projecting this surface on the base.

5. A circle with radius a is projected into an ellipse with semi-axes a and b ; show by projection that the area of the ellipse is πab .

6. The vertical angle of a conical tent is 67° , and the radius of the base is $5\frac{1}{2}$ feet; find (i) the slant height, (ii) the area of canvas used, (iii) the content of the tent.

7. A pyramid on a square base is such that each of the other faces is an isosceles right-angled triangle, find by projection the angle between a triangular face and the base.

Geometrical representation of imaginary quantities.

78. In Fig. LII OA is of length r .

By the usual convention a line OA drawn to the right is

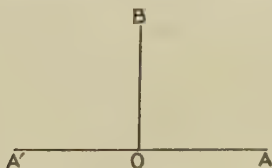


Fig. LII.

considered positive, so that (OA) represents $+r$. If now (OA) is turned through two right angles, it takes up the position (OA') and, by the usual convention, (OA') represents $-r$. Hence the

geometrical operation of turning through two right angles represents the algebraical operation of multiplying by -1 . Let us consider what the operation of turning through one right angle represents.

This is an operation which, if performed twice in succession, turns through two right angles, which represents multiplication by -1 .

But the algebraical operation of multiplying by $\sqrt{-1}$, if performed twice in succession, multiplies by -1 .

Hence it seems reasonable that the operation of turning a vector line through a right angle represents the algebraical operation of multiplying by $\sqrt{-1}$; that is, (OB) at right angles to (OA) represents $r \times \sqrt{-1}$, i.e. $\sqrt{-1}r$.

In future we shall denote $\sqrt{-1}$ by i .

79. With the interpretation of i suggested by the last section,

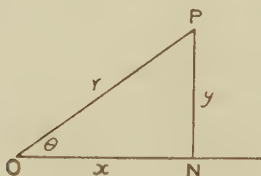


Fig. LIII.

$x + iy$ is represented by a vector line of length x followed by a vector line of length y at right angles to the first vector.

$$\begin{aligned} x + iy &= (ON) + (NP) && \text{(Fig. LIII.)} \\ &= (OP). && \text{(By vector addition, § 72.)} \end{aligned}$$

Or, in words, $x + iy$ is represented by the vector (OP) , that is by a vector line of length $\sqrt{x^2 + y^2}$, making an angle $\tan^{-1} \frac{y}{x}$ with the positive direction.

80. For our purposes this statement is more useful if reversed, viz.

$$\begin{aligned} (OP) &= x + iy \\ &= r \cos \theta + ir \sin \theta \\ &= (\cos \theta + i \sin \theta) r. \end{aligned}$$

Or, in words, the vector line of length r , in direction θ , represents the magnitude r multiplied by $\cos \theta + i \sin \theta$. This gives the important result that

turning through an angle θ represents multiplication by

$$\cos \theta + i \sin \theta.$$

Hence

turning twice in succession through θ represents multiplication by $\cos \theta + i \sin \theta$ repeated twice;

i.e. turning through 2θ represents multiplication by

$$(\cos \theta + i \sin \theta)^2;$$

but turning through 2θ represents multiplication by

$$(\cos 2\theta + i \sin 2\theta).$$

Hence the suggested interpretation of $\sqrt{-1}$ or i , leads to the identity $(\cos 2\theta + i \sin 2\theta) \equiv (\cos \theta + i \sin \theta)^2$.

If this is verified by algebraic multiplication and by the use of the ordinary formulae for $\cos 2\theta$ and $\sin 2\theta$, it will be found correct.

Carrying on the argument in the same way, we deduce that

$$(\cos n\theta + i \sin n\theta) \equiv (\cos \theta + i \sin \theta)^n,$$

where n is any positive integer.

Again, turning through a half θ is an operation which, if repeated, turns through θ , and, therefore, represents a multiplication which, if repeated, multiplies by $\cos \theta + i \sin \theta$;

$$\text{i.e.} \quad (\cos \tfrac{1}{2}\theta + i \sin \tfrac{1}{2}\theta) = (\cos \theta + i \sin \theta)^{\frac{1}{2}}.$$

$$\text{Similarly,} \quad \left(\cos \frac{\theta}{n} + i \sin \frac{\theta}{n} \right) = (\cos \theta + i \sin \theta)^{\frac{1}{n}};$$

$$\text{and} \quad \left(\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta \right) = (\cos \theta + i \sin \theta)^{\frac{p}{q}}.$$

Lastly, turning through $-\theta$ cancels turning through θ , and, therefore, represents an operation which cancels multiplication by $(\cos \theta + i \sin \theta)$;

$$\text{i.e.} \quad \{\cos(-\theta) + i \sin(-\theta)\} = (\cos \theta + i \sin \theta)^{-1}.$$

Similarly,

$$\cos(-n\theta) + i \sin(-n\theta) = (\cos \theta + i \sin \theta)^{-n},$$

where n is any positive quantity.

We have now deduced from the geometrical interpretation of $\sqrt{-1}$, that

$$(\cos n\theta + i \sin n\theta) = (\cos \theta + i \sin \theta)^n$$

for all real values of n .

This is known as **De Moivre's Theorem**.

Example. Use De Moivre's Theorem to find $\sqrt[3]{1}$.

$$\cos 2n\pi = 1, \quad \sin 2n\pi = 0,$$

where n is zero or any integer.

$$\text{Hence} \quad 1 = \cos 2n\pi + i \sin 2n\pi;$$

$$\begin{aligned} \therefore \sqrt[3]{1} &= (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{3}} \\ &= \cos \frac{2}{3}n\pi + i \sin \frac{2}{3}n\pi. \end{aligned}$$

$$\text{If } n = 0, \quad \sqrt[3]{1} = \cos 0 + i \sin 0 = 1.$$

$$\text{If } n = 1, \quad \sqrt[3]{1} = \cos \frac{2}{3}\pi + i \sin \frac{2}{3}\pi = \frac{1}{2}(-1 + i\sqrt{3}).$$

$$\text{If } n = 2, \quad \sqrt[3]{1} = \cos \frac{4}{3}\pi + i \sin \frac{4}{3}\pi = \frac{1}{2}(-1 - i\sqrt{3}).$$

$$\text{If } n = 3, \quad \sqrt[3]{1} = \cos 2\pi + i \sin 2\pi = 1.$$

For other values of n it is seen that the three roots are repeated. Hence De Moivre's Theorem shows that there are three different cube roots of unity. They are, of course, the three roots of the equation $x^3 - 1 = 0$. The student should verify that the same roots are obtained by Algebra.

Examples VIII d.

Represent graphically and by imaginary quantities the following vectors:

1. (i) Magnitude, $r = 25$; direction $\theta = \alpha$ where $\alpha = \tan^{-1} \frac{4}{3}$.

(ii) „ „ $r = 25$; „ $\theta = \pi - \alpha$;

(iii) „ „ $r = 25$; „ $\theta = \pi + \alpha$;

(iv) „ „ $r = 25$; „ $\theta = 2\pi - \alpha$;

(v) „ „ $r = 25$; „ $\theta = -\alpha$.

2. Show graphically that

$$(x' + iy') + (x'' - iy'') = (x' + x'') + i(y' + y'').$$

3. Express the following in the form $\gamma(\cos \theta + i \sin \theta)$:

(i) $3 + 4i$;

(ii) $5 + 6i$;

(iii) $7 - 8i$;

(iv) $-5 - 12i$;

(v) $-5 + 12i$;

(vi) $8i$.

4. Interpret geometrically $(\cos \alpha + i \sin \alpha)(\cos \alpha - i \sin \alpha)r$; and justify your interpretation.

5. Show graphically that

$$(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \cos (\alpha + \beta) + i \sin (\alpha + \beta).$$

6. Verify De Moivre's Theorem by calculation, when $n = 2, 3, -1, -2$.

7. Assuming De Moivre's Theorem, prove that $\sqrt[3]{-1}$ has three values, viz. $-1, \frac{1}{2}(1 + \sqrt{-3})$ and $\frac{1}{2}(1 - \sqrt{-3})$.

8. (a) Prove De Moivre's Theorem by induction when n is a positive integer.

(b) Deduce the proof when n is not a positive integer, by methods similar to those used for the Binomial Theorem in Algebra.*

Examples VIII.

1. A man walks one kilometre in a direction 16 degrees North of East; he then turns to the left, through an angle of 110 degrees, and walks one kilometre in the new direction. How far is he North and how far East of his starting-point?

2. Show that $a \cos \theta + b \sin \theta$ can be expressed in the form $r \cos (\theta - \alpha)$. Illustrate by a figure.

3. A number of rods are jointed together, and the two free ends are secured to two points A and B in the same horizontal line and distant c inches. If the length of the r^{th} rod is a_r , and its inclination to the horizontal is θ_r (all the angles being measured in the same sense), prove that (i) $\sum (a_r \cos \theta_r) = c$; (ii) $\sum (a_r \sin \theta_r) = 0$. (See § 89, Example ii.)

4. Prove by projection that

$$\sin (90 + A) = \cos A \text{ and } \sin (270 - A) = -\cos A.$$

5. In what respects can a vector quantity be represented by a straight line?

If three forces P, Q, R , acting at a point O , are such that $P/\sin QR = Q/\sin RP = R/\sin PQ$ (where $\sin PQ$ denotes the sine of the angle between P and Q), show that the three forces produce equilibrium.

6. A man walks one kilometre in a direction A° North of East, one kilometre in a direction making 120° with the first direction, and one kilometre at an angle 240° with the first direction. Draw a figure showing that he has now returned to the starting-

* See *School Algebra*, pp. 407, 435, 465.

point; and by considering the distances he has gone to the East and North write down two trigonometrical identities concerning the sines and cosines of A , $120 + A$, $240 + A$.

7. Suggest a geometrical construction which may help to sum the series :

(i) $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \dots n \text{ terms};$

(ii) $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \dots n \text{ terms}.$

Deduce that both these sums become zero if $n\beta = 2\pi$.

8. A body which weighs 12 lb. is kept at rest by means of two cords, one being horizontal and the other inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$; find the forces exerted by the cords.

9. A mine shaft is 1650 feet in length. It slopes downwards at an angle of 45° to the horizon for a certain part of its total length, say x feet, and at an angle of 35° for the rest of its length. If the total depth reached is 1000 feet, obtain an equation for x , and hence calculate x .

10. A man playing five holes of a golf course first walks 260 yards due East, then 140 yards 20° South of East, then 300 yards due South, then 290 yards 40° West of North, then 220 yards 30° West of South, thus arriving at the fifth hole. Find how far the fifth hole is from the first tee.

11. The perpendicular from the origin on a straight line equals p and makes an angle α with the axis of x ; by projecting the co-ordinates of any point on the line show that the equation of the straight line may be put in the form $x \cos \alpha + y \sin \alpha = p$.

(This is known as the perpendicular form of the equation of a straight line.)

Hence find the length of the perpendiculars from the origin on the lines whose equations are (i) $3x + 4y = 7$; (ii) $5x - 12y = 2$; (iii) $x + 2y = 6$. Verify by drawing to scale.

12. The co-ordinates of a point referred to rectangular axes OX , OY are x , y ; referred to two rectangular axes OV , OW through the same point O the co-ordinates are ξ , η . Prove by projection that $\xi = x \cos \alpha + y \sin \alpha$, where α is the angle between OX and OV .

Find three other similar relations connecting ξ , η , x , y .

CHAPTER IX

FORMULAE FOR (i) THE FUNCTIONS OF THE
SUM OR DIFFERENCE OF TWO ANGLES,
(ii) THE SUM OR DIFFERENCE OF THE
FUNCTIONS OF TWO ANGLES, (iii) THE FUNC-
TIONS OF THE DOUBLE ANGLE AND THE
HALF-ANGLE

81. To express $\cos (A + B)$ in terms of the sines and cosines of A and B .

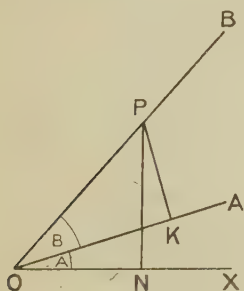


Fig. LIV.

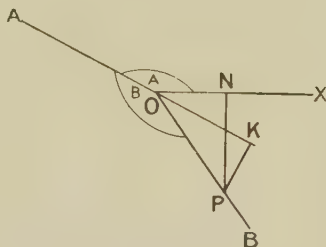


Fig. LV.

Let OX be the initial line ; and let the revolving line first turn through the angle A to the position OA and then continue to turn through an additional angle B to the position OB . Then OB is the bounding line of the angle $A + B$. Along OB measure a length $OP = r$ units.

Project OP on the initial line, produced backwards if necessary ; also project OP on OA , produced backwards if necessary.

Figures LIV and LV show two of the many possible cases.

In all cases

the projection of (OP) on OX

= sum of the projections of (OK) and (KP) on OX ;

i.e. $r \cos(A+B) = (OK) \cos A + (KP) \cos(A+90)$

$$= r \cos B \cos A + r \sin B (-\sin A).$$

Hence $\cos(A+B) = \cos A \cos B - \sin A \sin B$.*

Several proofs of this have already been given, but the earlier proofs have implied that A and B are together less than two right angles; this proof applies whatever be the values of A and B .

Exercises. (i) Deduce the formula for $\sin(A+B)$ by substituting $90-A$ in place of A , and $-B$ in place of B .

(ii) By similar substitutions deduce the formulae for $\cos(A-B)$ and $\sin(A-B)$.

(iii) By projecting perpendicular to OX , find the expanded form of $\sin(A+B)$.

(iv) Modify the construction so as to prove directly, by projection, the formula for $\sin(A-B)$ and $\cos(A-B)$.

(v) Complete the following formulae:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

(vi) Learn these formulae in words, as:

$$\cos \text{ sum} = \cos \cos - \sin \sin.$$

$$\cos \text{ diff} = \cos \cos + \sin \sin$$

$$\sin \text{ sum} = \sin \cos + \cos \sin$$

$$\sin \text{ diff} = \sin \cos - \cos \sin$$

* When Fig. LV is being used it must be recollected that (OK) is negative, and that its inclination to OX is XOA not XOK , see § 68. If (OK) is regarded as positive, its actual length is $-r \cos B$; but the angle is then XOK , the cosine of which is $-\cos A$. Whatever way it is taken, the projection of (OK) on OX is found to be $r \cos B \cos A$.

82. The following proof does not involve any knowledge of projection; its chief drawback is that it applies only to the case when $A+B$ is less than a right angle. It is easily modified to suit any other given case.

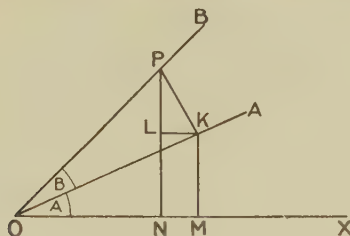


Fig. LVI.

Let $XOA = A$ and $AOB = B$;

then $XOB = A+B$, and in Fig. LVI is less than 90° .

Take a point P in the bounding line of $A+B$;

let fall PN perpendicular to OX ;

„ PK „ „ OA ;

„ KL „ „ PN and, therefore, parallel to OX ;

„ KM „ „ OB .

$$\begin{aligned}\cos(A+B) &= \frac{ON}{OP} = \frac{OM-MN}{OP} \\ &= \left[\frac{OM}{OP} - \frac{LK}{OP} \right] \\ &= \frac{OM}{OK} \frac{OK}{OP} - \frac{LK}{PK} \frac{PK}{OP}.\end{aligned}$$

The spaces in the bracketed line (which does not appear in the completed work) are filled in with the hypotenuses of the triangles in which the respective numerators occur.

Now angle $LPK = 90^\circ - LKP = LKO = A$; therefore

$$LK/PK = \sin A.$$

Hence $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

From the same figure, prove that

$$\sin(A+B) =$$

To find the functions of $A-B$, the angle AOB must be made on the negative side of OA . The point P must be taken in the

bounding line of $A - B$, and the construction and proof proceed as before. It is found that

$$\cos(A - B) =$$

$$\sin(A - B) =$$

Exercises. (i) Prove the four formulae when A and B are each less than 90° but $A + B$ is greater than 90° .

(Make the same construction as when $A + B$ is less than 90° , and pay careful attention to the signs of the lines.)

(ii) Prove the four formulae when A and B are each obtuse and together greater than 270° .

Examples IX a.

1. By using the formulae of § 81, verify that $\sin(90 - A) = \cos A$, $\cos(90 + A) = -\sin A$, $\sin(180 - A) = \sin A$, $\cos(180 - A) = -\cos A$, $\sin(270 + A) = -\cos A$, $\cos(360 - A) = \cos A$.

2. Express $\cos 70^\circ$ in terms of the functions of (i) 40° and 30° ; (ii) 45° and 25° ; (iii) 95° and 25° ; (iv) 35° .

3. Express $\sin 40^\circ$ in terms of (i) 30° and 10° ; (ii) 25° and 15° ; (iii) 70° and 30° ; (iv) 20° .

4. From the expansions of $\sin(A + B)$ and $\cos(A + B)$ deduce the expansion of $\tan(A + B)$ in terms of $\tan A$ and $\tan B$.

5. From the expansions of $\sin(A - B)$ and $\cos(A - B)$ deduce the expansion of $\tan(A - B)$ in terms of $\tan A$ and $\tan B$.

6. Verify that $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$ by using the formulae for $A - B$.

7. Show that (i) $\sin(A + B) \cos B - \cos(A + B) \sin B = \sin A$;

(ii) $\cos(A + B) \cos B + \sin(A + B) \sin B = \cos A$

8. From the formulae for $A + B$ deduce that

$$\sin 2A = 2 \sin A \cos A \text{ and } \cos 2A = \cos^2 A - \sin^2 A.$$

What is the value of $\tan 2A$?

9. Find the values of

(i) $\sin(A + B) + \sin(A - B)$; (ii) $\cos(A + B) + \cos(A - B)$;

(iii) $\sin(A + B) - \sin(A - B)$; (iv) $\cos(A + B) - \cos(A - B)$.

Account for the signs of (iii) and (iv) from first principles.

10. Prove that

$$\sin A = \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B).$$

Prove similar results for $\cos A$, $\sin B$, and $\cos B$.

11. From the results of 10 deduce that

$$\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B),$$

and three similar results.

12. Prove that

$$(i) \cos^2 \theta + \cos^2 \phi - 2 \cos \theta \cos \phi \cos (\theta + \phi) = \sin^2 (\theta + \phi);$$

$$(ii) \sin^2 \theta + \cos^2 \phi - 2 \sin \theta \cos \phi \sin (\theta + \phi) = \cos^2 (\theta + \phi).$$

83. Sums and differences of sines or cosines expressed as products.

These formulae are most easily derived from the formulae of § 81, as suggested in Examples IX *a*. They can be proved independently by projection.

Make the angle $XOA = A$ and the angle $XOB = B$.*

On OA and OB take lengths OP , OQ respectively, each equal to r units. Join PQ .

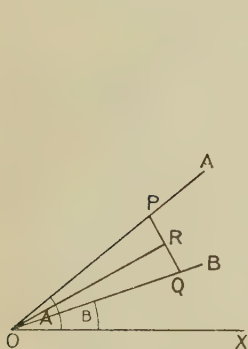


Fig. LVII.

Bisect the angle QOP by a line cutting PQ at R .

Then the angles ROP , ROQ each equal $\frac{1}{2}(A-B)$; and the angle $XOR = \frac{1}{2}(A+B)$.

From congruent triangles $RP = RQ$, and PRO is a right angle.

The projection of $(OP) =$ sum of projections of (OR) and (RP) , and the projection of $(OQ) =$ sum of projections of (OR) and (RQ) .

* Notice the difference between this construction and the construction of § 81.

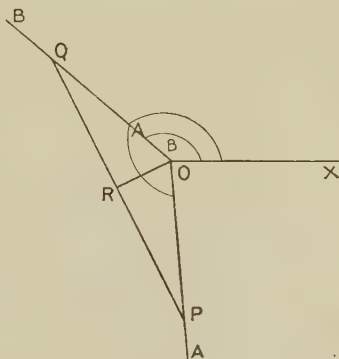


Fig. LVIII.

$\therefore$ projection of (OP) + projection of $(OQ) = 2^{\text{nd}}$ projection of (OR) , since projections of (RP) and (RQ) are equal but opposite.

Projecting on a line perpendicular to OX , we have

$$r \sin A + r \sin B = 2^{\text{nd}} \text{ the projection of } (OR).$$

But (OR) is the projection of (OP) on the direction OR ,

$$\text{i.e.} \quad (OR) = r \cos \frac{1}{2}(A - B).$$

$$\therefore \quad r \sin A + r \sin B = 2r \cos \frac{1}{2}(A - B) \sin \frac{1}{2}(A + B).$$

$$\text{Hence} \quad \sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$$

From the same figure, by projection on OX , we have

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$$

Again, projection of (OP) - projection of $(OQ) = 2^{\text{nd}}$ projection of (RP) .

$$\text{Hence} \quad \sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B),$$

$$\cos A - \cos B = -2 \sin \frac{1}{2}(A - B) \sin \frac{1}{2}(A + B).$$

The proofs apply to all cases whatever be the magnitudes of A and B .

The reason for the negative sign in the last of these formulae is obvious, for, if $A > B$, then $\cos A < \cos B$.

It is useful to learn the formulae in words, it being understood in all cases that the greater angle is put first.

$$\text{sine} + \text{sine} = 2 \text{ sine half sum } \cos \text{ half difference.}$$

$$\begin{aligned} &= \\ &= \\ &= \end{aligned}$$

84. Products of sines and cosines expressed as sums or differences.

In the formulae of the last section put

$$\frac{1}{2}(A + B) = X, \quad \frac{1}{2}(A - B) = Y;$$

so that $A = X + Y$, $B = X - Y$.

$$\text{Then} \quad \sin(X + Y) + \sin(X - Y) = 2 \sin X \cos Y,$$

$$\text{i.e.} \quad 2 \sin X \cos Y = \sin(X + Y) + \sin(X - Y).$$

Similar results are obtained from the other formulae. If we replace X by A and Y by B , the formulae become

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B),$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B),$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B),$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B).$$

These are more easily proved direct from the $A + B$ and $A - B$ formulae.

In using these formulae it is usual (but not necessary) to put the greater angle first; this shows why there are distinct formulae for $2 \sin A \cos B$ and $2 \cos A \sin B$.

Express the four formulae in words:

Twice sine cos = sin sum + sin difference.

=

=

=

Examples IX b.

1. Apply the formulae of §§ 83, 84 to the following cases and verify from the tables:

- (i) $A = 70^\circ, B = 30^\circ$; (iii) $A = 72^\circ, B = 18^\circ$;
 (ii) $A = 110^\circ, B = 75^\circ$; (iv) $A = 78^\circ, B = 46^\circ$.

2. Prove, from the formula for $\sin A + \sin B$, that

$$\sin 2\theta = 2 \sin \theta \cos \theta,$$

and, in a similar way, show that

$$1 + \cos A = 2 \cos^2 \frac{1}{2} A, \quad 1 - \cos A = 2 \sin^2 \frac{1}{2} A.$$

3. Prove that

$$(i) \sin A + \cos A = \sqrt{2} \cos (A - 45^\circ);$$

$$(ii) \sin A - \cos B = -2 \sin \left(45^\circ - \frac{A+B}{2} \right) \sin \left(45^\circ - \frac{A-B}{2} \right);$$

$$(iii) \cos A + \sin B = 2 \cos \left(45^\circ + \frac{A-B}{2} \right) \cos \left(45^\circ + \frac{A+B}{2} \right).$$

4. Prove that (i) $\sin 50^\circ + \sin 130^\circ = 2 \cos 40^\circ$;
 (ii) $(\cos 50^\circ - \cos 130^\circ) = 2 \sin 40^\circ$.

Verify these by squaring and adding.

- ✓ 5. Prove that (i) $2 \cos 40^\circ \sin 50^\circ = 1 - \sin 10^\circ$;
 (ii) $2 \cos 40^\circ \sin 40^\circ = \sin 80^\circ$;
 (iii) $2 \sin 64^\circ \sin 26^\circ = \cos 38^\circ$.

Verify this last result from the tables.

6. Fill in the right-hand side of the following:

- (i) $\sin 70^\circ + \sin 50^\circ =$ (ii) $\cos 30^\circ - \cos 110^\circ =$
 (iii) $2 \sin 75^\circ \cos 10^\circ =$ (iv) $\sin 37^\circ + \cos 24^\circ =$
 (v) $2 \cos 84^\circ \cos 72^\circ =$ (vi) $\cos 79^\circ - \cos 52^\circ =$
 (vii) $\sin 75^\circ - \sin 116^\circ =$ (viii) $2 \cos 80^\circ \cos 35^\circ =$
 (ix) $\cos 24^\circ - \sin 76^\circ =$ (x) $2 \sin 17^\circ \sin 48^\circ =$

$$\begin{array}{ll}
 \text{(xi)} \quad 2 \cos 73^\circ \sin 15^\circ = & \text{(xii)} \quad 2 \cos 14^\circ \cos 166^\circ = \\
 \text{(xiii)} \quad \cos \frac{1}{2}\pi - \cos \frac{1}{6}\pi = & \text{(xiv)} \quad \sin \frac{1}{4}\pi + \cos \frac{1}{4}\pi = \\
 \text{(xv)} \quad 2 \sin 43^\circ \cos 47^\circ = & \text{(xvi)} \quad 2 \cos 97^\circ \sin 46^\circ = \\
 \text{(xvii)} \quad \sin 81^\circ + \sin 10^\circ = & \text{(xviii)} \quad \sin 49^\circ - \sin 53^\circ = \\
 \text{(xix)} \quad 2 \sin 79^\circ \sin 15^\circ = & \text{(xx)} \quad \cos 43^\circ - \cos 216^\circ =
 \end{array}$$

7. Prove that $4 \cos (75^\circ + A) \sin (75^\circ - A) = 1 - 2 \sin 2A$.

Formulae for the double angle and half-angle.

85. It has already been shown in § 51 that

$$\begin{aligned}
 \sin 2A &= 2 \sin A \cos A; \\
 \cos 2A &= \cos^2 A - \sin^2 A \\
 &= 1 - 2 \sin^2 A \\
 &= 2 \cos^2 A - 1.
 \end{aligned}$$

The proof there given assumed that $2A$ is less than 180° .

If we put A instead of B in the $A+B$ formulae the same results are obtained; thus they are true for all values of A .

The results can easily be proved independently by projection; the proofs are the same as in § 81, A taking the place of B .

86. From the last section, by putting $\frac{1}{2}A$ in place of A , we have

$$\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A = \cos A;$$

$$\text{also} \quad \cos^2 \frac{1}{2}A + \sin^2 \frac{1}{2}A = 1.$$

$$\text{Add} \quad 2 \cos^2 \frac{1}{2}A = 1 + \cos A.$$

$$\text{Subtract} \quad 2 \sin^2 \frac{1}{2}A = 1 - \cos A.$$

$$\begin{aligned}
 \text{Hence} \quad \cos \frac{1}{2}A &= \pm \frac{1}{2}(\sqrt{1 + \cos A}); \\
 \sin \frac{1}{2}A &= \pm \frac{1}{2}(\sqrt{1 - \cos A}).
 \end{aligned}$$

If the value of A is given, there is no ambiguity of sign. If, for instance, $A = 140^\circ$, then $\frac{1}{2}A = 70^\circ$, and the sine and cosine are both positive; if $A = 264^\circ$, then $\frac{1}{2}A = 132^\circ$, and the sine is positive, the cosine negative.

If the value of $\cos A$ is given but not the value of A , the ambiguity cannot be removed. Suppose $\cos A = \frac{1}{2}$, then A may have any value of the form $360^\circ n \pm 60^\circ$. Hence $\frac{1}{2}A$ may have any value given by $180^\circ n \pm 30^\circ$. If we tabulate these values, we have

angle	cosine	sine
30°	$+\frac{1}{2}\sqrt{3}$	$+\frac{1}{2}$
150°	$-\frac{1}{2}\sqrt{3}$	$+\frac{1}{2}$
210°	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$
330°	$+\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$

87. Tangent formulae.

From the sine and cosine formulae the following tangent formulae are derived ; the proof of the first only is given :

$$\begin{aligned}\tan (A+B) &= \frac{\sin (A+B)}{\cos (A+B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B}. \quad (\text{By dividing throughout by } \cos A \cos B.)\end{aligned}$$

Similarly,

$$\begin{aligned}\tan (A-B) &= \\ \tan 2A &= \\ \tan \frac{1}{2} A &= \pm \sqrt{\frac{1 - \cos A}{1 + \cos A}} \\ &= \frac{1 - \cos A}{\sin A} \text{ or } \frac{\sin A}{1 + \cos A}. \quad (\text{By rationalizing.})\end{aligned}$$

There is no ambiguity of sign after rationalizing.

88. These may all be proved directly from the figures used for the sine and cosine formulae ; e.g. in Fig. LVI.

$$\begin{aligned}\tan (A+B) &= \frac{NP}{ON} \\ &= \frac{MK+LP}{OM-LK} \\ &= \frac{\frac{MK}{OM} + \frac{LP}{OM}}{1 - \frac{LK}{PK} \frac{PK}{OM}} \quad (\text{By dividing so as to make the first term in the denominator to be 1.})\end{aligned}$$

The triangles LPK , OKM are similar ;

$$\therefore \frac{LP}{OM} = \frac{PK}{OK} = \tan B. \quad \text{Hence } \tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Exercises. Prove that

- (i) $\tan (A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}$;
- (ii) $\tan A + \tan B = \frac{\sin (A+B)}{\cos A \cos B}$;
- (iii) $\tan A - \tan B = \frac{\sin (A-B)}{\cos A \cos B}$;

$$(iv) \cot A + \cot B = \frac{\sin(A+B)}{\sin A \sin B};$$

$$(v) \cot A - \cot B = -\frac{\sin(A-B)}{\sin A \sin B}.$$

Example. To prove that $\tan^{-1} \frac{1}{99} + \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{1}{70}$.

Let $A = \tan^{-1} \frac{1}{99}$, $B = \tan^{-1} \frac{1}{239}$;
so that $\tan A = \frac{1}{99}$, $\tan B = \frac{1}{239}$.

$$\begin{aligned} \tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= \frac{\frac{1}{99} + \frac{1}{239}}{1 - \frac{1}{99} \times \frac{1}{239}} \\ &= \frac{239 + 99}{23900 - 240} = \frac{338}{23660} \\ &= \frac{1}{70}, \end{aligned}$$

i.e. $\tan^{-1} \frac{1}{99} + \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{1}{70}$.

Examples IX c.

1. Prove that $(\sin A + \cos A)^2 = 1 + \sin 2A$,
and $(\sin A - \cos A)^2 = 1 - \sin 2A$.

2. Assuming the values of $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$, deduce $\sin 90^\circ$, $\cos 90^\circ$, $\tan 90^\circ$.

3. Find a formula for $\cot 2A$ in terms of $\cot A$.

4. Show that $\sin 3A = 3 \sin A - 4 \sin^3 A$. Explain how it is that there are three values of $\sin A$ when $\sin 3A$ is given.

5. Find the values of $\tan 22\frac{1}{2}^\circ$, $\tan 67\frac{1}{2}^\circ$, $\tan 157\frac{1}{2}^\circ$.

6. Prove that $2 \sin \frac{1}{2} A = \pm \sqrt{1 + \sin A} \pm \sqrt{1 - \sin A}$.

Find $\sin \frac{1}{2} A$ when $\sin A = \frac{1}{2}$. Illustrate by a figure.

7. Find $\cos \frac{1}{2} A$ when $\sin A = \frac{1}{2}$. Illustrate by a figure.

8. Prove that (i) $\sin 2A = 2 \tan A \div (1 + \tan^2 A)$;
(ii) $\cos 2A = (1 - \tan^2 A) \div (1 + \tan^2 A)$;
(iii) $\tan 2A = 2 \tan A \div (1 - \tan^2 A)$.

9. Prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$;
 $\cos 3A = 4 \cos^3 A - 3 \cos A$;
 $\tan 3A = (3 \tan A - \tan^3 A) \div (1 - 3 \tan^2 A)$.

10. Show that

$$(\cos A + \sin A)^3 + (\cos A - \sin A)^3 = 3 \cos A - \cos 3A.$$

11 (a). Show that $\sin \frac{1}{2}A + \cos \frac{1}{2}A = \pm \sqrt{1 + \sin A}$,

and $\sin \frac{1}{2}A - \cos \frac{1}{2}A = \pm \sqrt{1 - \sin A}$.

(b) Having given $4 \sin 54^\circ = \sqrt{5} + 1$, apply the formulae in (a) to find $\sin 27^\circ$ and $\cos 27^\circ$, explaining how the ambiguities of sign are cleared up.

(c) Show that $8(\sin^2 42^\circ - \cos^2 78^\circ) = \sqrt{5} + 1$.

12. Prove that

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

Deduce the formula for $\tan 3A$.

What can be deduced if $A+B+C$ equals (i) 180° , (ii) 90° ?

13. If $\tan A = \frac{1}{2}\frac{3}{7}$ and $\tan B = \frac{7}{20}$, show that

$$A+B = (4n+1)\frac{1}{4}\pi.$$

14. Show that

$$\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 4 \cos \theta \cos 2\theta \cos 4\theta.$$

15. Find all the solutions of the equation

$$\sin \theta \sin 3\theta = \sin 5\theta \sin 7\theta.$$

16. If $\tan A = \frac{1}{5}$, $\tan B = \frac{3}{7}$, $\tan C = \frac{5}{27}$, and each angle is acute, prove that $A+B+C = \frac{1}{4}\pi$.

17. If $\tan \theta = \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta$, show that

$$\tan 2\theta = (\sin \alpha \sin \beta) \div (\cos \alpha + \cos \beta).$$

18. (i) If $\theta = \tan^{-1} \frac{1}{3}$, find $\tan 2\theta$.

(ii) Show that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{1}{4}\pi$.

19. Prove that

$$\cos 2A - \cos 2B = 2(\cos^2 A - \cos^2 B) = 2(\sin^2 B - \sin^2 A).$$

20. Prove that

$$(i) \frac{1}{a+b \cos \theta} = \frac{\sec^2 \frac{1}{2}\theta}{(a+b) + (a-b) \tan^2 \frac{1}{2}\theta};$$

$$(ii) \frac{1}{a \cos \theta + b \sin \theta} = \frac{1 + \tan^2 \frac{1}{2}\theta}{a + 2b \tan \frac{1}{2}\theta - a \tan^2 \frac{1}{2}\theta}.$$

21. Solve the equations

$$(i) x^2 - \sqrt{2} \sin(\frac{1}{4}\pi + \alpha)x + \frac{1}{2} \sin 2\alpha = 0;$$

$$(ii) x^2 - 2 \cot 2\beta \cdot x - 1 = 0.$$

22. Solve for α and V the following equations:

$$2ag = V^2 \sin 2\alpha, \quad 2bg = V^2 \sin^2 \alpha,$$

23. A hemispherical shell of radius 16 inches rests with its rim on a horizontal table; a rod is hinged to a vertical wall, 25 inches from the centre of the shell, at a point 5 feet above the table. The rod is in the same vertical plane as the hinge and centre of the shell, and touches the shell. Find its inclination to the vertical.

Oral Examples.

- (a) (i) $\sin(P - Q) =$ (ii) $\cos X + \cos Y =$
 (iii) $\cos(90 - \frac{1}{2}A + B) =$ (iv) $\sin 270^\circ =$
 (v) $2 \sin \alpha \cos \beta =$ (vi) $\cos^2 \theta - \sin^2 \theta =$
 (vii) $\tan(A - B) =$ (viii) $\sin B - \sin C =$
 (ix) $\cos^2 45^\circ - \sin^2 45^\circ =$ (x) $\cos 2A =$
- (b) (i) $\cos(C + A) =$ (ii) $\sin B + \sin C =$
 (iii) $2 \sin \frac{1}{2}(B + C) \sin \frac{1}{2}(B - C) =$ (iv) $\cos \theta + \cos \phi =$
 (v) $2 \cos^2 \frac{1}{2}C - 1 =$ (vi) $\tan 2B =$
 (vii) $\sin(180 - B + C) =$ (viii) $\cos^2 75^\circ + \sin^2 75^\circ =$
 (ix) $\cos(\frac{1}{2}A + B + \frac{1}{2}A - B) =$ (x) $\sin(360 - 2C) =$
- (c) (i) $\sin 2B =$ (ii) $\sin(P + Q) =$
 (iii) $\cos(\pi - \alpha + \beta) =$ (iv) $2 \sin \frac{1}{2}C \cos \frac{1}{2}C =$
 (v) $1 - 2 \sin^2 B =$ (vi) $\cos^2 \frac{1}{2}C - \sin^2 \frac{1}{2}C =$
 (vii) $\cos^2 \frac{1}{2}C + \sin^2 \frac{1}{2}C =$ (viii) $\tan(B - C) =$
 (ix) $\cos C - \cos A =$ (x) $(\sin B + \cos B)^2 =$
- (d) (i) $\cos B \cos C - \sin B \sin C =$ (ii) $\cos(X - Y) =$
 (iii) $\sin 3A =$ (iv) $\sin \frac{1}{2}B =$
 (v) $\cos B + \cos C =$ (vi) $2 \cos B \cos C =$
 (vii) $\sin^2(B + C) + \cos^2(B + C) =$ (viii) $2 \cos^2 \frac{1}{2}\theta - 1 =$
 (ix) $(\cos \frac{1}{2}A - \sin \frac{1}{2}A)^2 =$ (x) $\tan(90 - C) =$

89. The preceding formulae lead to a number of useful identities in the cases where $A + B + C = 90^\circ$ or 180° . The method of dealing with these is shown in the following illustrative examples.

Example i. In any triangle $\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2} A$.

[Here $\frac{b-c}{b+c}$ gives the clue to the proof.]

By the sine formula, $\frac{b}{\sin B} = \frac{c}{\sin C}$,

$$\text{i.e.} \quad \frac{\sin B}{\sin C} = \frac{b}{c},$$

Componendo et dividendo, $\frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c}$;

$$\therefore \quad \frac{2 \cos \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C)}{2 \sin \frac{1}{2}(B+C) \cos \frac{1}{2}(B-C)} = \frac{b-c}{b+c},$$

$$\text{i.e.} \quad \frac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)} = \frac{b-c}{b+c};$$

but $A+B+C = 180^\circ$; $\therefore \frac{1}{2}A + \frac{1}{2}(B+C) = 90$.

Hence $\tan \frac{1}{2}(B+C) = \tan (90 - \frac{1}{2}A) = \cot \frac{1}{2}A$.

Substituting above, $\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2}A$.

This formula has been proved geometrically in § 53; it is usually proved by the method given above.

Example ii. In any triangle

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C.$$

$$\begin{aligned} \text{L.H.S.} &= \cos A + \cos B + \cos C \\ &= 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) + 1 - 2 \sin^2 \frac{1}{2} C \\ &= 2 \sin \frac{1}{2} C \cos \frac{1}{2}(A-B) + 1 - 2 \sin \frac{1}{2} C \cos \frac{1}{2}(A+B), \\ &\quad \text{since } \frac{1}{2} C = 90 - \frac{1}{2}(A+B) \\ &= 1 + 2 \sin \frac{1}{2} C (\cos \frac{1}{2} A - \cos \frac{1}{2} A + \cos \frac{1}{2} B) \\ &= 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C. \end{aligned}$$

The symbol $\Sigma \cos A$ is sometimes used to denote

$$\cos A + \cos B + \cos C;$$

and $\Pi \sin A$ to denote $\sin A \sin B \sin C$. The above result can be written:

$$\Sigma \cos A = 1 + 4 \Pi \sin \frac{1}{2} A.$$

Example iii. In any triangle $\Sigma \cos^2 A = 1 - 2 \Pi \cos A$.

(Questions involving the sum of the squares of sines or cosines are usually solved by expressing these squares in terms of the cosine of the double angle.)

$$\begin{aligned} 2 \Sigma \cos^2 A &= 2 \cos^2 A + 2 \cos^2 B + 2 \cos^2 C \\ &= 1 + \cos 2A + 1 + \cos 2B + 2 \cos^2 C. \end{aligned}$$

(Note that one angle is left unchanged.)

$$\begin{aligned} \therefore \Sigma \cos^2 A &= 1 + \cos(A+B) \cos(A-B) + \cos^2 C \\ &= 1 - \cos C \cos(A-B) - \cos C \cos(A+B), \\ &\qquad\qquad\qquad \text{since } C = 180 - (A+B) \\ &= 1 - \cos C [\cos(A-B) - \cos(A+B)] \\ &= 1 - 2 \cos A \cos B \cos C. \end{aligned}$$

Example iv. Solve the equation

$$\sin \theta + \sin 2\theta + \sin 3\theta + \sin 4\theta = 0.$$

Rearrange $\sin \theta + \sin 4\theta + \sin 2\theta + \sin 3\theta = 0.$

Use formula for sum of two sines

$$\begin{aligned} 2 \sin \frac{5}{2}\theta \cos \frac{3}{2}\theta + 2 \sin \frac{5}{2}\theta \cos \frac{1}{2}\theta &= 0; \\ \therefore \text{either } \sin \frac{5}{2}\theta = 0 \text{ or } \cos \frac{3}{2}\theta + \cos \frac{1}{2}\theta &= 0; \\ \text{i.e. } \frac{5}{2}\theta = n\pi \text{ or } 2 \cos \theta \cos \frac{1}{2}\theta &= 0; \\ \text{i.e. } \theta = \frac{2}{5}n\pi, \text{ or } \cos \theta = 0 \text{ or } \cos \frac{1}{2}\theta &= 0; \\ \text{i.e. } \theta = (2n+1)\frac{1}{2}\pi \text{ or } \frac{1}{2}\theta = (2n+1)\frac{1}{2}\pi. \end{aligned}$$

Hence the complete solution is

$$\theta = (2n+1)\pi, (2n+1)\frac{1}{2}\pi \text{ or } \frac{2}{5}n\pi.$$

Example v. To prove that

$$r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$$

From the figure of § 59,

$$\begin{aligned} r \cot \frac{1}{2}B + r \cot \frac{1}{2}C &= a, \\ \text{i.e. } r \left(\frac{\cos \frac{1}{2}B}{\sin \frac{1}{2}B} + \frac{\cos \frac{1}{2}C}{\sin \frac{1}{2}C} \right) &= 2R \sin A, \end{aligned}$$

$$\text{i.e. } r \frac{\sin \frac{1}{2}B \cos \frac{1}{2}C + \cos \frac{1}{2}B \sin \frac{1}{2}C}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = 2R \sin A,$$

$$\text{i.e. } r \frac{\sin \frac{1}{2}(B+C)}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = 2R \sin A,$$

$$\text{i.e. } r \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}B \sin \frac{1}{2}C} = 4R \sin \frac{1}{2}A \cos \frac{1}{2}A;$$

$$\text{since } \frac{1}{2}(B+C) = 90 - \frac{1}{2}A;$$

$$\therefore r = 4R \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C.$$

Exercise. Prove that $r_1 = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

Example vi. To show that the distance between the circumcentre and in-centre = $\sqrt{R^2 - 2Rr}$.

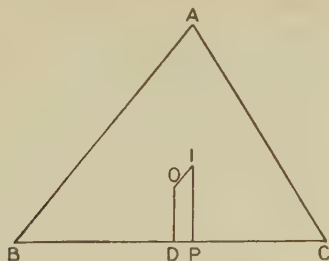


Fig. LIX.

In Fig. LIX, with the usual notation,

$$BD = R \sin A, \quad DO = R \cos A,$$

$$BP = r \cot \frac{1}{2} B, \quad PI = r.$$

$$\begin{aligned} OI^2 &= (BP - BD)^2 + (IP - OD)^2 \\ &= (r \cot \frac{1}{2} B - R \sin A)^2 + (r - R \cos A)^2 \\ &= R^2 - 2Rr(\sin A \cot \frac{1}{2} B + \cos A) + r^2(1 + \cot^2 \frac{1}{2} B) \\ &= R^2 - 2Rr \frac{\sin(A + \frac{1}{2} B)}{\sin \frac{1}{2} B} + r^2 \frac{1}{\sin^2 \frac{1}{2} B} \\ &= R^2 - 2Rr \frac{\sin(A + \frac{1}{2} B)}{\sin \frac{1}{2} B} + 4Rr \frac{\sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C}{\sin^2 \frac{1}{2} B} \quad \text{(Substituting for } r.) \\ &= R^2 - 2Rr \frac{\cos \frac{1}{2}(A - C) - 2 \sin \frac{1}{2} A \sin \frac{1}{2} C}{\sin \frac{1}{2} B}, \\ &\qquad\qquad\qquad \text{since } A + \frac{1}{2} B = 90 + \frac{1}{2} A - \frac{1}{2} C \\ &= R^2 - 2Rr \frac{\cos \frac{1}{2}(A + C)}{\sin \frac{1}{2} B}. \\ &= R^2 - 2Rr. \qquad\qquad\qquad \text{since } \frac{1}{2}(A + C) = 90 - \frac{1}{2} B. \end{aligned}$$

This is more shortly proved by Pure Geometry; but the method used here is a general method to find the lengths of lines connected with the triangle.

Example vii. To prove that

$$\begin{aligned} \sin A + \sin(A + B) + \sin(A + 2B) + \dots \text{ to } n \text{ terms} \\ = \frac{\sin \frac{1}{2} nB \sin(A + \frac{1}{2} n - 1 B)}{\sin \frac{1}{2} B}. \end{aligned}$$

Let $S = \sin A + \sin(A+B) + \dots + \sin(A + \overline{n-1} B)$.

Multiply by $2\sin\frac{1}{2}B$.

Then $2\sin\frac{1}{2}B \cdot S$

$$= 2\sin A \sin\frac{1}{2}B + 2\sin(A+B)\sin\frac{1}{2}B + \dots + 2\sin(A + n-1 B)\sin\frac{1}{2}B.$$

Use the formula for the product of two sines.

$$\begin{aligned} 2\sin\frac{1}{2}B \cdot S &= \cos(A - \frac{1}{2}B) - \cos(A + \frac{1}{2}B) \\ &\quad + \cos(A + \frac{1}{2}B) - \cos(A + \frac{3}{2}B) \\ &\quad + \dots - \dots \\ &\quad + \cos(A + \frac{1}{2}\overline{2n-3}B) - \cos(A + \frac{1}{2}\overline{2n-1}B) \\ &= \cos(A - \frac{1}{2}B) - \cos(A + \frac{1}{2}\overline{2n-1}B); \end{aligned}$$

$$\therefore S = \frac{\sin\frac{1}{2}nB \sin(A + \frac{1}{2}\overline{n-1}B)}{\sin\frac{1}{2}B}.$$

Note. Compare this with the formula for the sum of n terms of an Arithmetic Progression. Notice that $A + \frac{1}{2}\overline{n-1}B$ = half the sum of the first angle (A) and the last angle ($A + \overline{n-1}B$).

Examples IX.

1. Prove the following identities:

- (i) $\sin 3A = 4 \sin A \sin(60^\circ + A) \sin(60^\circ - A)$;
- (ii) $\sin 3A \sin^3 A + \cos 3A \cos^3 A = \cos^3 2A$;
- (iii) $(1 - 2 \sin^2 A) \div (1 + \sin 2A) = (1 - \tan A) \div (1 + \tan A)$;
- (iv) $\frac{\tan(45^\circ + A) + \tan(45^\circ - A)}{\tan(45^\circ + A) - \tan(45^\circ - A)} = \operatorname{cosec} 2A$;
- (v) $\sin(y+z-x) + \sin(z+x-y) + \sin(x+y-z) - \sin(x+y+z)$
 $= 4 \sin x \sin y \sin z$;
- (vi) $\cot \frac{1}{4}\theta - \cot \theta = \operatorname{cosec} \theta + \operatorname{cosec} \frac{1}{2}\theta$;
- (vii) $\cos 4A + 2(\cos A + \sin A)^4 = 3 + 4 \sin 2A$;
- (viii) $\sin A + \sin B = \sin(A+B) + 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}(A+B)$;
- (ix) $\sin A - 3 \sin 3A + 3 \sin 5A - \sin 7A = 8 \sin^3 A \cos 4A$;
- (x) $\cos(A+B+C)$
 $= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C$
 $\quad - \sin A \sin B \cos C$;
- (xi) $\cos \frac{1}{2}A (2 \sin A - \sin 2A) = \sin^2 \frac{1}{2}A (2 \sin A + \sin 2A)$;
- (xii) $\cos A + \cos B + \cos C + \cos(A+B+C)$
 $= 4 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(B+C) \cos \frac{1}{2}(C+A)$.

2. If A, B, C be the angles of a triangle, show that

$$(i) \tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C;$$

$$(ii) \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C;$$

$$(iii) \sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C + 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = 1;$$

$$(iv) \sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C;$$

$$(v) \cot A \cot B + \cot A \cot C + \cot B \cot C = 1;$$

$$(vi) \cot A + \cot B + \cot C \\ = \cot A \cot B \cot C + \operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C;$$

$$(vii) \tan B \tan C + \tan C \tan A + \tan A \tan B \\ = 1 + \sec A \sec B \sec C;$$

$$(viii) \cos A \sin (B - C) + \cos B \sin (C - A) + \cos C \sin (A - B) = 0;$$

$$(ix) (\tan A + \tan B) (\tan A - \cot C) = \sec^2 A;$$

$$(x) \tan \frac{1}{2}B \tan \frac{1}{2}C + \tan \frac{1}{2}C \tan \frac{1}{2}A + \tan \frac{1}{2}A \tan \frac{1}{2}B = 1.$$

3. Show geometrically that $\sin (A + B) = \sin A \cos B + \cos A \sin B$ when each of the angles A and B is between $\frac{1}{2}\pi$ and π , and $A + B$ is less than $\frac{3}{2}\pi$.

4. Solve the equation $\cos 3A + \cos 2A + \cos A = 0$.

5. Find all the values of θ which satisfy

$$(i) \cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0;$$

$$(ii) \sin 3\theta + \sin 4\theta + \sin 5\theta = 0.$$

6. Solve (i) $\sin (A + 30^\circ) = 1 \div \sqrt{2}$;

$$(ii) \sqrt{3} \sin A + \cos A = \sqrt{2};$$

$$(iii) \sin A + \cos A = 1;$$

$$(iv) \sin A + \sqrt{3} \cos A = 2;$$

$$(v) \sqrt{2} (\cos 3x + \sin 3x) = 1;$$

$$(vi) a \cos \theta + b \sin \theta = c \text{ (put } a = r \cos \alpha, b = r \sin \alpha).$$

7. Prove that (i) $2 \sin^{-1} \frac{1}{2} \sqrt{2} = 90^\circ$;

$$(ii) 2 \tan^{-1} \frac{1}{2} = \tan^{-1} \frac{4}{3}.$$

8. In any triangle show that

$$R (\sin 2A + \sin 2B + \sin 2C) = 2r (\sin A + \sin B + \sin C).$$

9. In any triangle show that

$$a^2 \cos 2B + b^2 \cos 2A = a^2 + b^2 - 4ab \sin A \sin B.$$

10. Prove the formula $(b + c) \tan \frac{1}{2}(B - C) = (b - c) \cot \frac{1}{2}A$.

Write down two corresponding formulae.

11. Using the fact that $3 \times 18^\circ = 90^\circ - 2 \times 18^\circ$, find the values of $\sin 18^\circ$ and $\cos 18^\circ$.

Give a geometrical method for determining $\sin 18^\circ$.

12. Simplify

$$(i) \left(\frac{\sin 4A}{\sin A} - \frac{\cos 4A}{\cos A} \right) \div (\cot A + \cot 2A);$$

$$(ii) \frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} + \frac{2}{\sin 2\theta} + \frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta}.$$

13. D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides; P is the orthocentre. Prove that

- (i) $AP = 2R \cos A$; (ii) $PD = 2R \cos B \cos C$;
 (iii) perimeter of triangle $DEF = 4R \sin A \sin B \sin C$.

14. State the general formula for all angles having a given cosine.

Solve $\sin 3A + \sin 5A + \sin 7A = 0$.

15. Find $\sec(A+B)$ in terms of the secant and cosecant of A and B , and prove

$$\sec 105^\circ = \sqrt{2}(1 + \sqrt{3}).$$

16. Prove that

$$\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1); \text{ and that } \sin^2 30^\circ = \sin 18^\circ \sin 54^\circ.$$

Show that in any circle the chord of an arc of 108° is equal to the sum of the chords of arcs of 36° and 60° .

17. Given $\cos A = .28$, determine the value of $\tan \frac{1}{2}A$, and explain fully the reason of the ambiguity which presents itself in your result.

18. Prove that

$$\cos^{-1}x + \cos^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}),$$

and solve the equation

$$\tan^{-1}\{(x+1) \div (x-1)\} + \tan^{-1}\{(x-1) \div x\} = \tan^{-1}(-7).$$

19. Express $\sin 3A \div (\sin 2A - \sin A)$ in terms of $\cos A$.

20. Prove the identities:

$$(i) (1 + \cos A) \tan^2 \frac{1}{2}A = 1 - \cos A;$$

$$(ii) (\sec A + 2 \sin A)(\operatorname{cosec} A - 2 \cos A) = 2 \cos 2A \cot 2A.$$

21. In any triangle prove that $(b-c) \cos \frac{1}{2}A = a \sin \frac{1}{2}(B-C)$.

If $A = 80^\circ$, $a = 10$, $b-c = 5$, find B and C .

22. Prove the identity $\cos 2x \sin 3x = \sin x \cos 4x + \cos x \sin 2x$.

23. Solve the equations $\cos 2\theta = \cos(\theta - \alpha)$; $\cos 3\theta = \sin(\theta - \beta)$.

24. (i) If the equation of a straight line is put in the form $y = mx + c$, what is the geometrical interpretation of m ?

(ii) Show how to find the angle between two lines whose equations are $y = mx + c$, $y = m'x + c'$.

(iii) Deduce that the lines are at right angles if $mm' = -1$; and parallel if $m - m' = 0$.

(iv) Prove that the lines whose equations are $ax + by + c = 0$, $a'x + b'y + c' = 0$, are perpendicular if $aa' + bb' = 0$, and parallel if $a/a' = b/b'$.

25. Find the angle between the lines whose equations are

(i) $3x - 4y = 5$, $4x - 2y = 7$;

(ii) $4x + 3y = 6$, $3x - 4y = 9$;

(iii) $2x - y = 3$, $4x + 5y = 1$;

(iv) $2x - y = 3$, $4x + 2y = 5$;

(v) $2x + 4y = 5$, $x + 2y = 3$.

In each case verify by drawing to scale.

26. Find the equations of the straight lines drawn through the point $(3, 5)$, and respectively parallel and perpendicular to the line whose equation is $3x - 4y = 5$.

27. Find the equation of the straight line, parallel to the line whose equation is $x \cos \alpha + y \sin \alpha = p$, and passing through the point (x', y') . Deduce that the length of the perpendicular from (x', y') to the line $x \cos \alpha + y \sin \alpha = p$ is $x' \cos \alpha + y' \sin \alpha - p$.

28. Find in its simplest form the equation of the line joining the points $\{a \cos (\alpha + \beta), b \sin (\alpha + \beta)\}$, $\{a \cos (\alpha - \beta), b \sin (\alpha - \beta)\}$.

29. Prove that $\sin 55^\circ \sin 15^\circ - \sin 50^\circ \sin 10^\circ - \sin 65^\circ \sin 5^\circ = 0$.

30. Show that in any triangle

$$\frac{a^2 \sin (B - C)}{b + c} + \frac{b^2 \sin (C - A)}{c + a} + \frac{c^2 \sin (A - B)}{a + b} = 0.$$

31. If $2 \cos \theta = x + 1/x$ and $2 \cos \phi = y + 1/y$, prove that

$$2 \cos (\theta + \phi) = xy + 1/xy \text{ and } 2 \cos (\theta - \phi) = x/y + y/x.$$

32. If $\theta + \phi = 240^\circ$, and $\text{versin } \theta = 4 \text{versin } \phi$, find the values of θ and ϕ .

33. Draw a curve to represent the variations in sign and magnitude of $(\sin \theta - \sqrt{3} \cos \theta) \div (\sqrt{3} \sin \theta + \cos \theta)$, from $\theta = 0$ to $\theta = \pi$.

34. If α and β are the roots of $a \sin \theta + b \cos \theta + c = 0$, prove that

$$\frac{\cos \frac{1}{2}(\alpha + \beta)}{b} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{-c} = \frac{\sin \frac{1}{2}(\alpha + \beta)}{a}.$$

35. Eliminate θ and ϕ from

$$a \sin \theta + b \sin \phi = h, \quad (\text{i})$$

$$a \cos \theta - b \cos \phi = k, \quad (\text{ii})$$

$$\cos (\theta + \phi) = l. \quad (\text{iii})$$

36. Eliminate θ and ϕ when two equations are the same as (i) and (ii) in Ex. 35, and the third equation is (i) $\sin(\theta + \phi) = l$, (ii) $\tan(\theta + \phi) = l$.

37. Eliminate θ and ϕ from the equations

$$\frac{a}{b} = \frac{\cos(\phi + \alpha)}{\cos(\theta - \alpha)} = \frac{\sin \phi}{\sin \theta}, \quad a \sin(\theta - \alpha) + b \sin(\phi + \alpha) = c.$$

38. Expand $\sin 5\theta$ in terms of $\sin \theta$, and $\cos 6\theta$ in terms of $\cos \theta$.

39. If $\sin B$ is the arithmetic mean between $\sin A$ and $\cos A$, prove that $\cos 2B = \cos^2(A + 45^\circ)$.

40. If $a \cos \theta + b \sin \theta = c$, show that

$$\theta = \tan^{-1} b/a + \cos^{-1} c/(\sqrt{a^2 + b^2}).$$

41. Find the maximum and minimum values of

$$a \cos \theta + b \sin \theta = c.$$

Verify your answer when $a = 3$, $b = 5$, by drawing a graph.

42. Prove that

(i) $\sin \theta + \sin 2\theta + \sin 3\theta + \dots$ to n terms

$$= \frac{\sin \frac{1}{2}n\theta \sin \frac{1}{2}(n+1)\theta}{\sin \frac{1}{2}\theta};$$

(ii) $\cos A + \cos(A + B) + \cos(A + 2B) + \dots$ to n terms

$$= \frac{\sin \frac{1}{2}nB \cos(A + \frac{1}{2}(n-1)B)}{\sin \frac{1}{2}B};$$

(iii) $\cos \alpha + \cos(\alpha - \beta) + \cos(\alpha + 2\beta) + \dots$ to n terms $= 0$, if

$$n\beta = 2\pi.$$

43. Find the sum of n terms in the following series:

(i) $\sin^2 A + \sin^2(A + B) + \sin^2(A + 2B) + \dots$;

(ii) $\cos^2 A + \cos^2(A + B) + \cos^2(A + 2B) + \dots$;

(iii) $\sin A \sin 2A + \sin 2A \sin 3A + \sin 3A \sin 4A + \dots$

Revision Examples C.

(All the following examples are taken from recent Examination Papers.)

1. Find, without reference to the tables, the values of (i) $\sin 45^\circ$; (ii) $\cos 150^\circ$; (iii) the tangent of the obtuse angle whose sine is $1/\sqrt{10}$.

2. Trace the graph of the function $\cos \theta + 2\sin \theta$ between the values 0 and 180° of θ , and determine from your figure the value of θ for which the function (i) is greatest, (ii) is decreasing most rapidly.

3. Express $\tan \theta$ in terms of $\sec \theta$.

Show that $(\sin \theta - \cos \theta)(\sec \theta + \operatorname{cosec} \theta) = \tan \theta - \cot \theta$.

4. Prove that the sines of the angles of a triangle are in the ratios of the sides opposite them.

5. Solve the equation $2\cos x + \sin x = 2$.

6. In a right-angled triangle ACB , C being the right angle, the angle A is 35° , the side AB is 10 inches; find the other sides.

7. If $\cos(A+B) = \cos A \cos B - \sin A \sin B$, calculate $\cos(A+B)$ when $A = 50^\circ$ and $B = 50^\circ$.

8. If α is measured in radians,

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \dots$$

where $\underline{5}$ means $1 \times 2 \times 3 \times 4 \times 5$. Find $\sin \alpha$ correct to four significant figures when $\alpha = 0.3$. What is the angle α in degrees?

9. Define the tangent of an angle in such a way that your definition is true for all angles.

If θ be an acute angle, prove that $\cos(90^\circ + \theta) = -\sin \theta$.

10. Arrange in order of magnitude the angles

$$2\sin^{-1} 0.51, \frac{1}{2}\cos^{-1} 0.32, \tan^{-1} 8.9.$$

11. Draw the graph of $\cos x$ for values of x lying between 0° and 90° .

Use your figure to solve roughly the equation $x = 100 \cos x^\circ$, and verify your solution by the tables.

12. Given that $\sin 20^\circ = 0.34$ and $\cos 20^\circ = 0.94$, write down the values of $\sin 160^\circ + \cos 160^\circ$, of $\sin 250^\circ + \cos 250^\circ$, and of $\sin 340^\circ + \cos 340^\circ$.

13. In any triangle ABC , show that

$$(i) \quad c = a \cos B + b \cos A;$$

$$(ii) \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

Find c when $a = 5$, $b = 6$, and $C = 155^\circ 31'$, having given $\cos 24^\circ 29' = 0.91$. Verify your result by a diagram drawn to scale.

14. Find to the nearest degree the angle subtended at a man's eye by a tower 50 feet high, when the man has stepped back 30 feet from the tower, assuming the height of his eye above the ground to be 5 feet 6 inches.

15. Write down a formula for $\sin \frac{1}{2}A$ in terms of the sides of the triangle ABC and explain the notation. How is the formula modified when $b = c$?

Given that the sides are 100, 200, 160 units in length, calculate the smallest angle.

16. A and B are two acute angles but $A + B$ is obtuse; prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

Solve completely $\cos x + \sin x = \cos \alpha - \sin \alpha$.

17. Define the tangent of an angle, and show geometrically that $\tan A \tan(90^\circ + A) + 1 = 0$.

18. Draw a circle of diameter 1 inch. Draw a diameter AB and the tangent to the circle at B , divide either of the semi-circumferences between A and B into 8 equal parts, join A to the points of section, and produce the joining lines to meet the tangent at B . Measure the distances of the points so found from B , and use the results obtained for drawing the graph of $\tan A$ from $A = 0^\circ$ to $A = 90^\circ$.

19. Prove that the area of the triangle ABC is

$$\frac{1}{2}a^2 \sin B \sin C / \sin A.$$

Use this expression to find the area of the triangle when

$$a = 106.5 \text{ yards, } A = 56^\circ 37', B = 75^\circ 46'.$$

20. A person walking along a straight level road running due East and West observes that two objects P and Q are in a line bearing North-West, and after walking a further distance d he observes that P bears due North and that the direction of Q makes an angle A with the direction in which he is walking. Prove that the distance PQ is $d \cos A / \sin(A - 45^\circ)$. Find PQ when $d = 1372$ yards, and the angle $A = 56^\circ 31'$.

21. (i) Show that $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.

(ii) Considering only values of A between 0° and 90° , find the value of A when $\sin A \cos A$ has its greatest value, and show that the same value of A gives the greatest value of $\sin A + \cos A$.

22. Let AD bisect the angle A of a triangle ABC , and let it meet BC in D ; show that $BD \sin B = CD \sin C$.

Hence show that $BD \cdot AC = DC \cdot AB$.

23. (i) Show geometrically that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

when A , B , and $A + B$ are each less than 90° .

(ii) By means of this formula, and in view of the restrictions under which it has been obtained, show that

$$\sin 464^\circ = \sin 153^\circ \cos 311^\circ + \cos 153^\circ \sin 311^\circ.$$

24. Find $\tan \theta$ and x in terms of a and b from the equations

$$\begin{aligned} a \sin \theta + b \cos \theta &= 3x, \\ a \cos \theta - 2b \sin \theta &= 2x. \end{aligned}$$

25. An angle is made to increase gradually from 0° to 360° ; state briefly how the values of its sine and of its cosine change during the increase of the angle.

26. Calculate the values of A between 0° and 360° for which $\tan A - 2 \cot A = 1$.

27. A and B are two milestones on a straight road running due East across a horizontal plane, C an object on the plane. The bearings of C as viewed from A and B are 35° North of East, and 55° North of West respectively. Find, to the nearest foot, (1) the distance of C from A , (2) the distance of C from the nearest point of the road.

28. Plot in relation to the same axis and origin the values of $\tan x$ and $2 \sin x$ for the values 0° , $12^\circ 30'$, $37^\circ 30'$, 50° , $62^\circ 30'$, 75° of x , draw the graphs of $\tan x$ and $2 \sin x$, and find from them the values of x for which $\tan x = 2 \sin x$. Give the general solution of the equation $\tan x = 2 \sin x$.

29. Prove that $(\cos A + \sin A) \div (\cos A - \sin A) = \tan (A + 45^\circ)$.

30. Prove for a triangle in which the angle B is obtuse the relation $\sin B/b = \sin C/c$, and deduce the relation

$$\tan \frac{1}{2}(B - C) = (b - c)/(b + c) \cot \frac{1}{2} A.$$

If $b = 27.3$ yards, $c = 15.8$ yards, $A = 48^\circ 36'$, find B and C .

31. Prove that in a triangle $r = 4R \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$. ABC is a triangle; $B'C'$ is drawn through A parallel to BC , $A'C'$ through C perpendicular to AB , and $A'B'$ through B perpendicular to AC . Prove that the area of the triangle $A'B'C'$ is

$$\frac{1}{2} a^2 \cos^2 (B - C) \div \cos B \cos C \sin A.$$

32. (i) Find $\sin A + \sin B$ in terms of functions of half the sum and of half the difference of the angles A and B .

(ii) If $A + B$ is between 90° and 180° , find under what circumstances $\tan A + \tan B$ will be negative.

33. Find to the nearest minute the angle of a regular polygon of 17 sides.

What angles less than 360° satisfy the equation

$$2 \cos^2 \theta + 11 \sin \theta - 7 = 0?$$

34. Prove the identity

$$\frac{\tan^3 A}{1 + \tan^2 A} + \frac{\cot^3 A}{1 + \cot^2 A} = \frac{1 - 2 \sin^2 A \cos^2 A}{\sin A \cos A}.$$

35. Assuming the formula $a^2 = b^2 + c^2 - 2bc \cos A$, establish a formula for $\tan \frac{1}{2} A$ in terms of the sides of the triangle, and find the greatest angle of the triangle whose sides are 13, 14, 15.

36. Prove that for any triangle ABC

$$a/\sin A = b/\sin B = c/\sin C.$$

If $B = 39^\circ 17'$, $a = 4.2$, and $b = 3.5$, solve the triangle fully; draw a figure to illustrate your solution.

37. The angles of elevation of a vertical pole from two points on a horizontal line passing through its base and 6 feet apart are α and β ; prove that the height of the pole is $6/(\cot \alpha - \cot \beta)$ feet.

38. From a point on a horizontal plane passing through the foot of a tower the angles of elevation of the top and bottom of a flagstaff 20 feet high, placed vertically at the summit of the tower, are $51\cdot2^\circ$ and $47\cdot3^\circ$. Find the height of the tower.

39. Prove that (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$;

(ii) $\cos 2A(1 + \tan^2 A) = 1 - \tan^2 A$.

Use (ii) to find the value of $\tan 15^\circ$.

40. Reduce the fraction $a \div (\cos^2 A - \sin^2 B)$ to a form suitable for logarithmic calculation, and perform the calculation when $a = 10$, $A = 29^\circ 55'$, and $B = 15^\circ 5'$.

41. Prove that $\sin^2 A + \cos^2 A = 1$ for all values of A less than 180° .

A and B are each less than 180° , $\sin A = \cdot 3900$, $\sin B = \cdot 9208$, find four possible values of $A+B$.

42. Find from your tables the value to two decimal places of the expression $\sin \theta + \sin 2\theta$, when θ is 10° , 20° , 30° , ... 90° , and from these draw a graph of the expression on a suitable scale.

43. In a triangle ABC prove that

(i) $2bc \cos A = b^2 + c^2 - a^2$;

(ii) $\cos^2 A + \cos^2 B + \cos A \cos B = \frac{3}{4}$, if $C = 60^\circ$.

44. In a triangle $a = 12\cdot76$, $b = 10\cdot87$, $c = 8\cdot37$, find C .

45. Show how to find the distance between two visible but inaccessible objects.

46. In any triangle ABC show that four times the area equals $(a^2 + b^2 + c^2) \div (\cotan A + \cotan B + \cotan C)$.

Show also that when C is a right angle this expression reduces to $2ab$.

47. Prove the identities:

(i) $1/\sin 2A = 1/\tan A - 1/\tan 2A = \tan A + 1/\tan 2A$;

(ii) $\tan 3A = \frac{\sin 2A + \sin 4A}{\cos 2A + \cos 4A}$.

48. What is the meaning of $\tan^{-1} x$?

Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1} [(x+y) \div (1-xy)]$.

Prove that 45° is one value of $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{8}$.

49. Prove that (i) $\sec^2 A = 1 + \tan^2 A$;

(ii) $\operatorname{cosec} \theta - \cot \theta = \tan \frac{1}{2} \theta$.

50. Construct an angle whose sine is 0.76. From your figure obtain the value of the cosine of the angle.

51. On squared paper draw graphs of $\tan \theta$ and $\cot \theta$ between $\theta = 10^\circ$ and $\theta = 80^\circ$. From the graph, or otherwise, find angles which satisfy the equation $\tan \theta + \cot \theta = 3$.

52. Let D be the point in which one of the escribed circles touches the side BC of a triangle ABC . If the sides a, b, c of the triangle are given, find expressions for the radius of that circle and for BD and CD .

53. A tree which grows at a point A on the north bank of a river is observed from the points B and C on the south bank. The distance BC is 200 metres, the angle ABC is $46^\circ 30'$, and the angle ACB is $58^\circ 20'$. Calculate the distance of A from the straight line BC .

54. Prove the formula $\sin \frac{1}{2} A = \sqrt{(s-b)(s-c) \div bc}$.

If, in a triangle ABC , $2b = a + c$, prove that

$$\sin \frac{1}{2} B = 2 \sin \frac{1}{2} A \sin \frac{1}{2} C.$$

55. Find the angles B and C and the radius of the circumscribed circle of a triangle ABC in which $A = 32^\circ 42'$, $a = 36$, $b = 44$.

56. State De Moivre's Theorem, and, assuming it for integral indices, prove it for fractional indices.

Write down all the values of $(\sqrt{-1})^{\frac{2}{3}}$.

57. If A is an obtuse angle whose sine is $\frac{5}{13}$, find the values of $\cos A$ and $\tan A$.

58. (i) Show, by drawing graphs of the two expressions $\sin x$ and $\cos(x + 90^\circ)$, that $\sin x = -\cos(x + 90^\circ)$.

(ii) If $\sin x = \frac{1}{2} \sqrt{2}$, find a formula which gives all the values of x which satisfy the equation.

59. Prove that in a triangle

(i) $\tan \frac{1}{2} B = \sqrt{(s-c)(s-a) \div s(s-b)}$;

(ii) $b \cos B + c \cos C = a \cos(B - C)$.

60. If two sides of a triangle and the angle opposite one of them are given, show how to solve the triangle, and discuss by the aid of a figure all the cases that can arise.

One side of a triangle is 20 inches long, the opposite angle is $34^{\circ}42'$; another side is $30\cdot41$ inches. Find the sides and angles of the two possible triangles.

61. Assuming the formulae for the sine and cosine of half an angle of a triangle in terms of the sides, prove that

$$(i) r = \sqrt{(s-a)(s-b)(s-c) \div s};$$

$$(ii) R = a/2 \sin A.$$

62. I observe the altitude of an airship to be 35° , and that of the sun, which is in the same vertical plane as my eye and the airship, to be 40° . The shadow of the airship falls on a tree on the same level as my eye and 500 feet in front of me. Find the height of the airship.

63. In any triangle prove that

$$\sin A - \sin B + \sin C = 4 \sin \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C.$$

Assuming the formula for expanding $\tan(A+B)$, find expressions for $\tan 2A$ and $\tan 3A$ in terms of $\tan A$.

64. Make an angle AOC and bisect it by the line OB . From any point A in OA draw ABC perpendicular to OB , meeting OB , OC in the points B and C respectively, and draw AN perpendicular to OC . Use this figure to prove that

$$(i) \sin 2A < 2 \sin A; \quad (ii) \tan 2A > 2 \tan A.$$

65. Prove that $\sin^2 A + \cos^2 A = 1$.

Having given that the sine of an angle is $\cdot56$, calculate its cosine.

66. Show how to construct an angle whose sine is $\cdot6$.

Find a value of x which satisfies the equation

$$4 \sin x + 3 \cos x = 1.$$

67. Given two sides of a triangle and the included angle, show how to find the remaining side and the other angles. Prove such formulae as you require.

If $a = 1097$ feet, $b = 781$ feet, $C = 31^{\circ}30'$, find c to the nearest foot.

68. A ship is sailing at the rate of 7 miles an hour. A man walks forward across the deck at the rate of 4 miles an hour

relative to the deck, in a direction inclined to the keel at an angle of 60° . Find the direction of his actual motion in space.

69. Prove the formula $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

Show that if $xy = a^2 + 1$ then

$$\cot^{-1}(a+x) + \cot^{-1}(a+y) = \cot^{-1}a.$$

70. Find an expression for $\cos(\alpha + \beta + \gamma)$ in terms of sines and cosines of α , β , and γ .

Prove the identity

$$\begin{aligned} \cos \alpha \cos(\beta + \gamma) + \cos \beta \cos(\gamma + \alpha) + \cos \gamma \cos(\alpha + \beta) \\ = \cos(\alpha + \beta + \gamma) + 2 \cos \alpha \cos \beta \cos \gamma. \end{aligned}$$

71. At what angle must forces of 4 dynes and 5 dynes act so that their resultant may be a force of 6 dynes?

72. If θ be the circular measure of an angle, prove that, as θ is indefinitely diminished, the ratios $\theta : \sin \theta$, $\theta : \tan \theta$ approach to the limit unity.

A man standing beside one milestone on a straight road observes that the foot of the next milestone is on a level with his eyes, and that its height subtends an angle of $2' 55''$. Find the approximate height of that milestone.

73. Write down the values of $\sin 36^\circ$ and $\cos 36^\circ$ as given by your tables. Calculate the sum of the squares of these numbers to six decimal places, and explain why the result differs from unity.

74. Give definitions of the tangent and cotangent of an angle which is greater than 90° and less than 180° .

Prove that (i) $\tan(180 - \theta) = -\tan \theta$;

(ii) $\tan(90 + \theta) = -\cot \theta$.

75. In any triangle prove that $a/\sin A = b/\sin B = c/\sin C$.

If BC be 25 inches, and CA be 30 inches, and if the angle ABC be twice the angle CAB , find the angles of the triangle ABC , and show that the length of the third side is 11 inches.

76. P , Q , R are three villages. P lies 7 miles to the North-East of Q , and Q lies $11\frac{1}{4}$ miles to the North-West of R . Find the distance and bearing of P from R .

77. A point is moving with velocity 50 feet per second in a direction 60° North of East. Find the resolved parts of the velocity in directions East and North.

78. A man has before him on a level plane a conical hill of vertical angle 90° . Stationing himself at some distance from its foot he observes the angle of elevation α of an object which he knows to be half-way up to the summit. Show that the part of the hill above the object subtends at his eye an angle

$$\tan^{-1} \frac{\tan \alpha (1 - \tan \alpha)}{1 + \tan \alpha (1 + 2 \tan \alpha)}.$$

79. The latitude of London is 51°N. , and the radius of the Earth 4000 miles. How far is London from the Equator measured along the Earth's surface, and how far from the Earth's axis?

80. Prove that $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

Show that $\sin 10^\circ + \sin 20^\circ + \sin 40^\circ + \sin 50^\circ = \sin 70^\circ + \sin 80^\circ$.

$$\begin{array}{r} .9811 \\ + 2 \\ \hline 2.9811 \\ + 19022 \\ \hline 399462 \end{array}$$

$$\begin{array}{r} 76 \\ 97771 \\ \hline 832 \\ 832 \\ 832 \\ 832 \\ \hline 22271 \\ \hline 81241 \end{array}$$

MISCELLANEOUS PROBLEMS

(The following examples are taken from recent Army Entrance and Civil Service Papers.)

1. I take measurements to determine the air space of a rectangular hall: length 18·4 metres, breadth 11·8 metres, inclination to floor of diagonal of side wall $31^{\circ}8'$, of diagonal of end wall 44° . Calculate the air space.

More measurements were taken than were necessary. Check the measurements by deducing one of them from the other three.

2. The ancient Greeks measured the latitude of a place by setting up a vertical rod and comparing its length with the length of its shadow. Supposing observations taken at mid-day at the equinox (when the sun is vertical at the equator) to give $\frac{5}{3}$ as the ratio of the rod to shadow at Alexandria, and $\frac{4}{3}$ as the ratio at Carthage, find the latitude of each place.

3. The following method of determining the horizontal distance PR , and the difference of level QR between two points P and Q , is often used. A rod with fixed marks A, B on it is held vertical at Q , and the elevations of these points, viz. $ACD = \alpha$, $BCD = \beta$, are read by a telescope and divided circle at C , the axis of the telescope being a distance $CP = a$ above the ground at P . If $QA = b$, and $AB = s$, write down expressions for PR and QR . Find PR and QR when $\alpha = 6^{\circ}10'$, $\beta = 7^{\circ}36'$, the values of a, b , and s being 5 feet, $2\frac{1}{2}$ feet, and 5 feet respectively.

4. Three balls, 5 cm. in diameter, lie on a floor in contact, and a fourth equal ball is placed on them. Find the height of the centre of the fourth ball above the plane of the other three centres. Find also the inclination to the vertical of any line that touches both the top ball and one of the lower balls.

5. The curved surface of a right circular cone whose semi-vertical angle is 45° is made by cutting out a sector from a circular sheet of copper, the diameter of the sheet being 56 cm. Determine the angle of the required sector.

6. If tangents be drawn to the inscribed circle of a triangle parallel to the sides of the triangle, show that the areas of the triangles cut off by these tangents are inversely proportional to the areas of the corresponding escribed circles.

7. A rod BC , of length 5.8 cm., rotates about B . Another rod CA , of length 8.6 cm., has one end C hinged to the first rod, while the other end A slides along the line BO . By drawing the rods in various positions, find how the length of BA varies as the angle B increases; and show BA as a function of angle B in a graph for one revolution of BC , showing the actual length of BA and representing 30° by 1 cm.

Write down an equation connecting the angle B and the lengths of the three sides of the triangle ABC . Solve the equation to find the length of BA when angle B is 35° .

8. The extreme range of the guns of a fort is 8000 metres. A ship, 14000 metres distant, sailing due East at 24 kilometres an hour, notices the bearings of the fort to be $20^\circ 30'$ North of East. Find, to the nearest minute, when the ship will first come within range of the guns.

9. The face of a building is 136 feet long. A photographer wants to take the building from a point at which the face subtends an angle of 37° , and for this purpose he starts off from one corner of the building in a direction making an angle of 127° with the face in question. Find by calculation the distance from the corner at which he must take the photograph. Calculate the area of ground in the triangular space between his position and the face of the building.

10. From the top of a telephone pole three wires radiate in a horizontal plane. One wire, A , exerts a tension of 100 lb. weight; the next, B , makes an angle of 90° with A and exerts a tension of 80 lb. weight; the third, C , makes an angle of 35° with B and an angle of 125° with A , and exerts a tension of 90 lb. weight. It is required to equilibrate the three tensions by means of a fourth wire. Find its direction and tension.

11. A man passing along a straight road measures the angle between the direction of his advance and a line drawn to a house on his left. At a certain moment the angle is $36^\circ 21'$. He walks on 1500 yards and finds that the angle between the same direction

and the line to the house is now $125^{\circ} 36'$. Find the distance of the house from the road.

12. Plot a curve giving the sum of $4 \sin \theta$ and $3 \sin 2\theta$ from $\theta = 0^{\circ}$ to $\theta = 180^{\circ}$, and read off the angles at which the greatest and the least values respectively of this sum occur. For the angle use 1 cm. to represent 10 degrees, and for $4 \sin \theta + 3 \sin 2\theta$ use 1 cm. to represent unity. Also estimate the slope of the curve when $\theta = 90^{\circ}$ and when $\theta = 135^{\circ}$.

13. A , B , and C are three buoys marking the corners of a triangular yacht racecourse round an island. The angles A , B , and C of the triangle ABC are found to be 75° , 63° , and 42° respectively. P is a flagstaff on the island, from which A and B can be seen, and the distances of P from A and B are found by a range-finder to be 650 yards and 535 yards respectively, and the angle APB to be 137° . Calculate the length of one lap of the course.

14. Draw an angle XOP of 30° , making OP 2" long: through P draw PQ parallel to OX and in the same direction: produce XO to X' , making $OX' = OP$, and join $X'P$: cut off $PQ = PX'$. Join OQ and measure the angle XOQ carefully. Now denote XOQ by ϕ , XOP by θ , and OP by e , and write down an expression for the length PQ . Deduce an equation for θ and ϕ , and solve it for $\tan \phi$.

Use your tables to evaluate ϕ when $\theta = 30^{\circ}$, and compare your result with the measured value. It is said that the given construction trisects an angle. What is the percentage error for 30° ?

15. In running a survey the lengths of a series of lines are measured, and the angle each line makes with the direction of magnetic North is measured by a theodolite. The data booked are given in the table below:—

Line.	Length in feet.	Bearing.
AB	433	$29^{\circ} 15'$
BC	521	$89^{\circ} 12'$
CD	352	$132^{\circ} 38'$
DE	417	$233^{\circ} 25'$

The angles are measured clockwise from the magnetic North direction.

By an error the measured length of the closing line EA of the survey was not recorded, nor its bearing; from the data given in the table calculate these missing data.

16. AOB and COD are two straight roads crossing one another at an angle of 57° . A motor-car, travelling at the rate of 18 miles an hour along AOB , is 1500 yards from O , when a man, walking at the rate of three miles an hour along COD , is a quarter of a mile from O ; car and man are both approaching O . Find graphically the motion of the car relative to the man. Hence find the least distance between the car and the man, and when they are at this distance from one another.

17. In a triangle $a = 10$ cm., $b = 7$ cm., one angle is 95° . There being no restriction as to which angle of the triangle is 95° , discuss how many distinct triangles can be made. Select any one case, and for this case calculate the remaining sides and angles.

18. X and Y are two fixed points in a straight line, P a point which so moves that $\cos PXY + \cos PYX = k$ (a constant). Prove the accuracy of the following construction for obtaining the locus of P : With X and Y as centres describe circles of radius XY/k . From any point N in XY draw NAB perpendicular to XY cutting the former circle in A and the latter in B . Draw XA and YB , intersecting in P . Then P is a point on the locus.

19. A candle, C , is placed on the floor at a distance r from a point O on a wall, and at the same level as the candle-flame, and the angle which OC makes with a perpendicular to the wall at O , is θ . The illumination received on the wall at O from the candle is known to be equal to $A \cos \theta / r^2$ where A is a constant. If the candle be moved about on the floor in such a way that this illumination remains constant, plot on a diagram the curve described by the candle-flame.

20. Two small islands are 5 miles apart, and there is known to be a rock distant 3 miles from each. A ship is in such a position that the islands subtend an angle of 66° at the ship. Calculate, to the nearest hundredth of a mile, her least possible distance from the rock.

21. Find by means of a graph two acute angles θ for which $5 \sin 2\theta = 3 \sin \theta + 2.5$.

Find also the greatest value of $5 \sin 2\theta - 3 \sin \theta$ when θ is an acute angle, and the angle to which this value corresponds.

22. The elevation of an aeroplane which is flying horizontally on a fixed course at a height of 150 feet is taken at two instants

at an interval of 20 secs. At the first observation the elevation is 10° and the bearing is due North, and at the second the elevation is $6\frac{1}{2}^\circ$ and the bearing is N. 35° E. Find the course and speed of the aeroplane.

23. The strength of an electric current C is obtained from the formula $C = k \tan \theta$ where θ is the angle read off in degrees on an instrument, and k is a constant. If an observer makes an error of $\delta\theta$ in reading the angle θ , prove that the value of C thus obtained will be wrong by an amount equal to $\frac{1}{90}\pi C \operatorname{cosec} 2\theta \delta\theta$. Hence find the error per cent. in C produced by making a mistake of $\frac{1}{10}$ degree when θ is 60° .

What value of θ is likely to produce the smallest error in the value of C ?

24. If P denote the pressure of wind in lb. per square foot on a plane surface at right angles to the direction of the wind, and p denote the normal pressure of wind in lb. per square foot on a plane surface inclined at an angle θ to the direction of the wind, the following formulae are used to determine the ratio $p:P$.

$$(i) \quad p/P = (\sin \theta)^{1.84 \cos \theta - 1};$$

$$(ii) \quad p/P = 2 \sin \theta / (1 + \sin^2 \theta).$$

Compare the values of p/P given by these formulae for the values 10° and 50° of θ .

25. A man walks due W. from a point A up a straight path inclined at 10° to the horizon. After walking 2 miles he reaches B , and turns up another straight path to the NE., sloping 15° upwards. He reaches C after walking one mile from B . What is the distance in a straight line from C to A ? What is the height of C above the level of A ? Taking the face of the hill ABC as a plane surface, what is the greatest slope?

26. A flagstaff stands vertically on horizontal ground. Four ropes, each 56 feet long, are stretched from a point in the flagstaff, 50 feet above the ground, to four pegs in the ground, arranged at the corners of a square. Calculate the angle between two adjoining ropes.

27. Q is the centre of a circle of radius 10 cm., and QO is a radius. The seven points $ABC\dots$ lie on the circumference and the angles $OQA, OQB, OQC\dots$ have the values $10^\circ, 20^\circ, 30^\circ\dots 70^\circ$. Find by drawing or calculation the lengths of the chords $OA, OB, OC\dots$, and tabulate the results.

Draw a graph to give the length of chord of the circle in terms of the angle which it subtends at the centre (for angles up to 70°). Show the chord's actual size, and represent 4 degrees by 1 cm.

From your graph find the length of the chord which subtends an angle of 48° . Make a triangle having one side of this length, and the other two sides 10 cm. long, and therefore having an angle of 48° .

Check the accuracy of your drawing by measuring this angle.

28. A square made of jointed rods each 4 inches long is deformed into a rhombus having half the area of the square. Calculate the lengths of the diagonals of the resulting figure and check by drawing. If it is part of a lattice-work, the original height of which is 6 times the diagonal of one of these squares, find by calculation how much the height of the lattice-work could be increased if each square were reduced to half its area.

29. A straight rod AB , 3 feet 9 inches long, is held under water, A being 2 feet 6 inches and B 9 inches below the surface. Calculate (a) the distance below the surface of a point C on the stick which is 12 inches from A , (b) the angle which the stick makes with the surface of the water.

If a parallelogram is held under water, show that in every position the sum of the depths of the 4 corners is 4 times the depth of the point of intersection of the diagonals.

30. If a closed loop of thread is placed on a soap-film that covers a ring of wire, and the film within the loop is pierced, the film outside takes up as small an area as possible and thus pulls the thread at A into a circle. Calculate the diameter and the area of the circle formed by the thread if length of thread forming the loop is 6 cm.

If the ends BC of the thread are attached to the ring, and the film on one side of the thread is pierced, the thread again becomes a circular arc. If the thread BC is 6 cm. long, and the angle it subtends at the centre of the circle of which it forms an arc is 120° , calculate the length of the chord BC .

EXAMINATION PAPERS

OXFORD AND CAMBRIDGE SCHOOLS' EXAMINATION BOARD.

SCHOOL CERTIFICATE, 1910.

1. Define the tangent of an angle.

Construct an acute angle whose sine is $\cdot 6$, and find its cosine and cotangent.

2. Prove that $\cos(180^\circ - \alpha) = -\cos \alpha$.

Arrange the angles α, β, γ in order of magnitude, if

$$\sin \alpha = \cdot 8211, \cos \beta = \cdot 7738, \tan \gamma = -0\cdot 6104,$$

the angles being positive and each less than 180° .

3. What is the length of the shadow of a man, 5 feet 8 inches high, cast by the sun when its altitude is $55^\circ 30'$?

4. Draw the graph of $10 + 10 \cos 2x$ for values of x between 0° and 60° . Find a value of x to satisfy the equation

$$x = 10 + 10 \cos 2x^\circ.$$

[Take one-tenth of an inch as unit along both axes.]

5. Prove that in *any* triangle $\sin A/a = \sin B/b$.

If $A = 63^\circ$, $B = 49^\circ$, $a = 50$ inches, find b to the nearest tenth of an inch.

6. Prove that

$$(i) \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} + \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{2}{1 - 2 \sin^2 \theta};$$

$$(ii) (\sec \theta + \tan \theta)(\operatorname{cosec} \theta - \cot \theta) = (\operatorname{cosec} \theta + 1)(\sec \theta - 1).$$

7. If $2 \sin \theta + 5 \cos \theta = 5$, prove that $\tan \theta = 0$ or $20/21$.

8. Prove that $\sin(A - B) = \sin A \cos B - \cos A \sin B$, where A and B are both acute angles and A is greater than B .

Prove that $\frac{\sin 5A + \sin A}{\sin 3A - \sin A} = 1 + 2 \cos 2A$.

9. Show that in any triangle ABC

$$\frac{b+c}{a} = \frac{\cos \frac{1}{2}(B-C)}{\sin \frac{1}{2}A}.$$

If $b+c = 24\cdot 8$ cm., $a = 11\cdot 89$ cm., $A = 39^\circ$, find B and C .

10. A lighthouse is observed from a ship which is steaming due N. to bear 62° W. of N.; after the ship has sailed 10 miles the lighthouse is observed to bear 40° W. of S. Calculate the distance of the ship from the lighthouse when it was nearest to it.

HIGHER CERTIFICATE, 1910.

PART I.

1. Give a definition of $\cos \theta$ that holds for all angles from 0° to 180° . Show that $\cos(180 - \theta) = -\cos \theta$.

2. Show that $\sec^2 \theta = 1 + \tan^2 \theta$.

Draw the graph of $1 + \sin 3x^\circ$, where x lies between 0° and 60° .

3. Construct an acute angle whose cotangent is 2, an obtuse angle whose sine is 3, and an obtuse angle whose secant is -3.5 . Measure these angles as accurately as you can with the protractor, and verify your results by means of tables.

4. (i) Verify that 30° , 45° , and 60° are solutions of the equation $\sin 3x + \cos 3x = 2 \cos 2x$.

(ii) Show that

$$(\operatorname{cosec} A + \sec A)^3 + (\operatorname{cosec} A - \sec A)^3 = 2 \operatorname{cosec}^3 A (3 \sec^2 A - 2).$$

5. Show that in an obtuse-angled triangle

$$\sin A/a = \sin B/b = \sin C/c.$$

A man observes that the angular elevation of the foot of a tower on a distant hillside is α , and that the angular elevation of the top of the tower is β , and he knows that the height of the tower is h feet. Show that his horizontal distance from the tower is $h \cos \alpha \cos \beta \operatorname{cosec}(\beta - \alpha)$.

PART II.

6. Draw the graph of $\cot x$ between the values -180 and $+180$ of x , taking the unit of x to be $\frac{1}{80}$ inch and the unit of y to be one inch.

Find an acute angle to satisfy the equation $x = 60 \cot x^\circ$.

7. Show that $\sin(A - B) = \sin A \cos B - \sin B \cos A$, taking A and B to be acute angles of which A is the greater.

If $\tan x = k \tan(A - x)$, show that

$$(k - 1) \sin A = (k + 1) \sin(2x - A).$$

Use this result and tables to solve the equation

$$\tan x = 2 \tan(50^\circ - x).$$

8. In the triangle in which $a = 72$ feet, $B = 40^\circ$, and $C = 55^\circ$, find c .

9. Find in terms of a , b , and c the radius of the circle escribed to the side BC of the triangle ABC .

If I_1 is the centre of this circle, show that

$$aAI_1^2 - bBI_1^2 - cCI_1^2 = abc.$$

10. AB is a diameter of a circle whose centre is O ; on AB an equilateral triangle ABC is described, and a point D is taken in AB such that $7BD = 2AB$; CD is produced beyond D to meet the circle at E . Show that $\tan ADC = 7/\sqrt{3}$ and that $\sin OED = 3/\sqrt{52}$.

Hence, or otherwise, show that the error made in taking the arc BE to be one-seventh of the circumference of the circle is less than '2 per cent.

Part III was beyond the scope of this book.

OXFORD LOCAL EXAMINATIONS.

JUNIOR. 1910.

1. (i) Find the sine of 60° ;

(ii) If A is an acute angle, and $\cos A = \frac{4}{5}$, find the value of $4 \tan A + 5 \sin A$.

2. P and Q are points on a straight stretch of a river bank and R is a point on the other bank. If $\cot PQR = \cdot 32$, $\cot QPR = \cdot 43$, and the length of PQ is 15 yards, find the breadth of the river.

3. Draw the graph of $\sin(45^\circ + 2x)$ between $x = 0$ and 180° .

4. If A , B , $A - B$ are all positive acute angles, prove that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

5. (i) A , B , C are the angles of a triangle; if $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the angle C .

(ii) Prove that $\frac{\cos 5A + \cos 3A}{\sin 5A - \sin 3A} = \cot A$.

6. Solve the equation $\cos 2\theta + \sin \theta = 1$.

7. Prove for any triangle that

(i) $a/\sin A = b/\sin B = c/\sin C$;

(ii) $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$.

8. Find the angles A and B of a triangle ABC in which $a = 13$, $b = 14$, $c = 15$, having given:

$$\log 2 = \cdot 3010, \log 7 = \cdot 8451,$$

$$\angle \tan 26^\circ 34' = 9\cdot 6990,$$

$$\angle \tan 29^\circ 44' = 9\cdot 7569.$$

SENIOR. 1910.

1. Find the tangent of 30° .

Using the values of $\tan 30^\circ$ and $\tan 45^\circ$, prove that

$$\tan 75^\circ = 2 + \sqrt{3}.$$

2. A man on a straight level road observed two objects P and Q (P being the nearer) in a horizontal straight line inclined to the direction of the road at an angle α . If $\tan \alpha = \cdot 75$, $PQ = 400$ yards, and the shortest distance of P from the road is 180 yards, what is the shortest distance of Q from the road?

3. Prove that $\cos 3A = 4\cos^3 A - 3\cos A$. Find $\sin 18^\circ$.

4. If ABC is a triangle in which $b = c = 5$ inches and $a = 8$ inches, find the values of $\tan A$ and $\tan B$.

5. Prove that

$$\cos^2 A + \cos^2 B = \sin^2 (A+B) + 2\cos A \cos B \cos (A+B).$$

6. Prove that in any triangle $c = (a+b)\sin \theta$, where

$$\cos \theta = 2\sqrt{ab} \cos \frac{1}{2}C / (a+b).$$

In a triangle ABC , $a = 36$ feet, $b = 4$ feet, $C = 55^\circ$. Using the above formula, find the third side, having given

$$\log 6 = \cdot 7782, \angle \cos 57^\circ 51' = 9\cdot 7261,$$

$$\angle \cos 27^\circ 30' = 9\cdot 9479, \sin 57^\circ 51' = \cdot 8467.$$

7. Find the radius of the circle inscribed in the triangle ABC .

C is the centre of a circle of diameter d , and A, B are two points on the circumference of the circle. If l is the length of the chord AB and δ is the diameter of the circle which touches CA , CB and also the arc AB at its middle point, prove that $1/\delta = 1/d + 1/l$.

CAMBRIDGE LOCAL EXAMINATION.

JUNIOR. 1909.

1. Define the sine of an angle. What are the greatest and least values which the sine of an angle can have?

Prove that $\sin A = \cos A \times \tan A$, and that

$$\sin A \sin B \cot B = \cos A \cos B \tan A.$$

2. Construct an angle whose tangent is 1.45, and measure it with a protractor. Verify your results with the help of the tables.

3. Prove that

$$(i) \sin A = \tan A / (\sqrt{1 + \tan^2 A}); \quad (ii) \cos(90^\circ + A) = -\sin A.$$

4. Find by drawing graphs of $\sin A$ and $\sin 2A$ for what value of A , less than 90° , $2 \sin A - \sin 2A = 1$.

5. A vertical post casts a shadow 15 feet long when the altitude of the sun is 50° ; calculate the length of the shadow when the altitude of the sun is 32° .

6. Prove that $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$, and that $\tan 2A = 2 \tan A / (1 - \tan^2 A)$.

Show that $\sin A - 3 \sin 3A + 3 \sin 5A - \sin 7A = 8 \sin^3 A \cos 4A$.

7. Prove that, in any triangle ABC , $a \cos B + b \cos A = c$.

Show also that $(\tan A + \tan B)(\tan A - \cot C) = \sec^2 A$.

8. Show how to solve a triangle when three sides are given.

Find the greatest angle of the triangle whose sides are 5.2 inches, 7.7 inches, and 9.1 inches.

SENIOR. 1909.

1. Show that the ratio of the circumference to the diameter of a circle is an invariable quantity.

Find to an inch the diameter of a wheel which makes 400 revolutions in rolling along a track one mile long.

2. Any positive proper fraction being given, show that there are two angles, one acute and the other obtuse, such that the sine of either is equal to this fraction.

If the fraction is $\frac{1}{3}$, use the tables to find the angles, and the cosine and tangent of each.

3. Find by aid of the tables the values of $\sin x - \tan 2x$ for the values $0^\circ, 10^\circ, 20^\circ, 30^\circ, 45^\circ, 60^\circ$ of x .

Make a graph to give the values of $\sin x - \tan 2x$ from $x = 0$ to $x = 60^\circ$.

4. Show that $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$.

Prove also that

$$(i) \tan^2 A = (1 - \cos 2A) \div (1 + \cos 2A);$$

$$(ii) \sin 55^\circ \sin 15^\circ - \sin 50^\circ \sin 10^\circ - \sin 65^\circ \sin 5^\circ = 0.$$

5. Find the greatest angle of a triangle whose sides are 15, 21, 28 inches in length.

Show that in any triangle

$$\frac{a^2 \sin(B-C)}{b+c} + \frac{b^2 \sin(C-A)}{c+a} + \frac{c^2 \sin(A-B)}{a+b} = 0.$$

6. Find an expression for the radius of the inscribed circle of a given triangle.

Determine to one place of decimals the length of the radii of the inscribed circle, and of the escribed circle opposite the greatest angle of the triangle referred to in Question 5.

Questions 7 and 8 were outside the scope of this book.

(The two following questions may be taken instead of 7 and 8, but considerably lower marks will be assigned to them.)

A. Show that if A, B, C are the angles of a triangle,

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Show also that

$$\tan \frac{1}{2} B \tan \frac{1}{2} C + \tan \frac{1}{2} C \tan \frac{1}{2} A + \tan \frac{1}{2} A \tan \frac{1}{2} B = 1.$$

B. Solve the equation $a \cos \theta + b \sin \theta = c$.

Find all the solutions of $\sin \theta \sin 3\theta = \sin 5\theta \sin 7\theta$.

COLLEGE OF PRECEPTORS.

CHRISTMAS, 1910.

$1\frac{1}{2}$ Hours.

[Four-place tables of logarithms and of natural functions and square-ruled paper are provided. All diagrams should be drawn as accurately as possible.]

PART I.

1. Define a radian, and find its magnitude in degrees to two places of decimals ($\pi = 2\frac{2}{7}$).

If the angle of an equilateral triangle were taken to be the unit angle, what would be the measure of a radian to two places of decimals?

2. Define the sine and tangent of an acute angle. Prove that

$$\sin^2 A + \cos^2 A = 1.$$

If $\tan A = \frac{15}{11\frac{1}{2}}$, find the value of $\cos A - 8 \sin A$.

3. Find, geometrically, $\tan 30^\circ$.

If $A = 30^\circ$, $B = 45^\circ$, $C = 60^\circ$, $D = 90^\circ$, find the value of:

- (i) $\sin A \cos B - \sin B \cos A$;
 (ii) $(\tan^2 B - \operatorname{cosec}^2 A) / (\cot C + \cos D)$.

4. Use logarithms to find as nearly as possible the values of:

- (i) $3.142 \times .9342 / .00532$; (ii) $\sqrt{562.3 / .005984}$.

5. Solve, using the tables, the triangle in which $C = 90^\circ$, $a = 654$, $A = 38^\circ 45'$.

PART II.

6. Find all the positive values of θ , less than 360° , which satisfy the equations:

- (i) $\cos^2 \theta - \sin^2 \theta = 0$;
 (ii) $4 \sin^2 \theta \cos^2 \theta - \sin^2 \theta = \frac{1}{2}$.

Which of the following statements are possible?

- (i) $\tan \theta = -2$; (ii) $\sin \theta = \frac{3}{2}$.

7. Write down, without proof, the expansions of $\sin(A - B)$, $\cos(A - B)$.

Find the value of $\tan \overline{A - B}$ in terms of $\tan A$, $\tan B$.

If $\tan A = \frac{4}{3}$, $\tan B = \frac{2}{7}$, find $\tan(A + B)$.

8. Prove that, in a triangle, $a^2 = b^2 + c^2 - 2bc \cos A$ when the angle A is (i) acute, (ii) obtuse.

Deduce that $\tan \frac{1}{2} A = \sqrt{(s-b)(s-c) \div s(s-a)}$.

Find the greatest angle in the triangle whose sides are 256, 389, 401.

9. AB is a horizontal straight line. A vertical straight line is drawn from B upwards, and in it two points P , Q are taken, such that BQ is five times BP . If the angle BAP is 30° , calculate $\tan PAQ$.

LEAVING CERTIFICATE EXAMINATION
(SCOTLAND). 1910.

1. Explain the circular measurement of angles.

Express 30° , 50° , $166^\circ 40'$ in radians.

Express $\cdot 0187$ radian in degrees, minutes, and seconds, taking $\pi = 3\cdot 1416$.

2. Taking a horizontal inch to represent 10° and 5 vertical inches to represent the unit of length, plot, with the help of your tables, the values of $\tan \theta$ when $\theta = 0, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ$.

Plot also the values of $\sin \theta$ for the same angles, join both series of points by smooth curves, and thus find a graphic solution of the equation $5(\tan \theta - \sin \theta) = 1$.

3. State the relation which exists between the sine and cosine of any angle.

Use this relation to find, and express in a diagram, all the values of α , less than 180° , which satisfy the equation

$$5 \sin \alpha + 6 \cos^2 \alpha = 7.$$

Either, 4 a. A man walked 5 miles due North and then walked 6 miles in a direction 27° East of North. Find by a figure drawn to scale how far he now is from his starting-point, and in what direction he should have originally started in order to go straight to his final position. Verify your results by calculation.

Or, 4 b. The sides of a parallelogram are 2 inches and 3 inches in length, and its area is $3\frac{1}{2}$ square inches. Find by a diagram the sizes of its angles and the length of its longer diagonal. Verify your results by calculation.

Either, 5 a. Draw a circle of radius 2 inches, and inscribe in it a triangle ABC , such that $\angle B = 34^\circ$, $\angle C = 73^\circ$.

Measure the lengths of the sides as nearly as possible.

Calculate with the help of the tables the lengths of the sides to the nearest hundredth of an inch, and thus test the correctness of your drawing.

Or, 5 b. State and prove the formula which gives $\tan(A+B)$ in terms of $\tan A$ and $\tan B$.

Apply this formula to find expressions for $\tan 2A$, $\tan 3A$, and $\tan 5A$ in terms of $\tan A$.

INTERMEDIATE EXAMINATION (IRELAND).

MIDDLE GRADE (PASS). 1910.

1. Prove that $\sin^2 A + \cos^2 A = 1$, where A is an obtuse angle.
2. Find the value of the expression $\operatorname{cosec} A - \frac{5}{6} \cot A$, if $\sin A = \frac{1}{6}$, when A is acute, and when A is obtuse.
3. Prove the identity $(1 - \tan^2 A) \div (2 \cos^2 A - 1) = \sec^2 A$.
4. In a triangle $C = 90^\circ$, $c = 65^\circ$, $\tan A = .28$. Find a and b each to two decimal places.
5. In a triangle $a = 5\sqrt{3}$, $b = 11$, $C = 150^\circ$. Find c and $\cos A$.
6. In a triangle $B = 45^\circ$, $b = 20$, $c = 4$. Find $\sin C$, and prove that the perpendicular from A on BC divides BC into two segments one of which is seven times the other.
7. Prove that the length of the perpendicular from the vertex A of a triangle on the opposite side BC is equal to $a(\cot B + \cot C)$, considering the cases when both angles are acute, when one is right, and when one is obtuse.
8. Find the angles between 0° and 360° which satisfy the equation $6 \sin \theta - 4 \operatorname{cosec} \theta + \cot \theta = 0$, being given $\cos 48^\circ 11' 23'' = \frac{2}{3}$.

MIDDLE GRADE (HONOURS). 1910.

1. Show by a graph the values of $\operatorname{cosec} A$ for values of A between -90° and 360° .
2. If A is an angle in the first quadrant, prove that
$$\sin A + \cos A + \tan A + \cot A > \sec A + \operatorname{cosec} A.$$
3. Prove the identity
$$3(\sin A - \cos A)^4 + 6(\sin A + \cos A)^2 + 4(\sin^6 A + \cos^6 A) = 13.$$
4. The sides of a triangle are 37, 7, and 40. Find all the angles, being given that $\cos 69^\circ 25' 48'' = \frac{1}{3}$.
5. In a triangle $a = \sqrt{5}$, $b = \sqrt{12}$, $C = 45^\circ$. Find c , and prove that $\cot A = 2\sqrt{\frac{6}{5}} - 1$.
6. Find a solution between 180° and 270° of the equation
$$5(1 + \sin x) = -3 \cos x,$$
 being given $\cos 28^\circ 4' 21'' = \frac{1}{17}$.

7. Prove by drawing a line through B , making an angle x with the side BC , or otherwise, that in a triangle ABC ,

$$c \cos (B-x) + b \cos (C+x) = a \cos x.$$

P is a point on the hypotenuse AB of a right-angled triangle ABC . $AP = x$, $PB = y$, $PC = z$. Find $\cos CPB$ in terms of x , y , and z . Find the sides of the triangle when $x = 3 - \sqrt{3}$, $y = \sqrt{3} + 1$, $z = \sqrt{6}$.

SENIOR GRADE (PASS). 1910.

1. Find the distance from the earth to the moon, assuming that the moon's diameter, 2165 miles, subtends an angle of $31' 10''$ at the earth.

2. Prove that $\tan \frac{1}{2} A = (1 - \cos A) / \sin A$.

Find $\tan 15^\circ$ and $\tan 22\frac{1}{2}^\circ$ without using the tables.

3. Find x if $\cos^{-1} x + \cot^{-1} 2 = \frac{1}{4}\pi$.

4. Assuming the formulae for the sines of the sum and difference of two angles, prove that

$$\sin A - \sin B = 2 \cos \frac{1}{2} (A+B) \sin \frac{1}{2} (A-B).$$

Find the corresponding expressions in factors for $\cos A - \cos B$.

5. Find the solutions between 0° and 360° of the equation

$$\cot 2x - 3 \tan x = 3.$$

6. In a triangle $a = 183$, $b = 247$, $C = 81^\circ 40'$. Find A and B .

7. In a triangle $A = 54^\circ 30'$, $B = 69^\circ 20'$, $a = 341$. Find b and c .

8. Prove that in a triangle

$$a \cos B - b \cos A = (a^2 - b^2) / c.$$

9. Prove that in a triangle $r \cot \frac{1}{2} A = s - a$, where r is the radius of the inscribed circle, and s the perimeter.

SENIOR GRADE (HONOURS). 1910.

1. An arc 40 feet in length is taken on a circle whose radius is 35 feet. Find, to the nearest inch, the length of the perpendicular from the centre on the chord of this arc.

2. Prove the identity

$$\cos 5A / \sin A + \sin 5A / \cos A = 2 \operatorname{cosec} 2A - 4 \sin 2A.$$

3. If $\cos x + \cos y + \cos z + \cos x \cos y \cos z = 0$, prove that

$$\tan \frac{1}{2} x \tan \frac{1}{2} y \tan \frac{1}{2} z = \pm 1.$$

4. If $x = \cot^{-1} \sqrt{\cos y} - \tan^{-1} \sqrt{\cos y}$, prove that

$$y = 2 \tan^{-1} \sqrt{\sin x}.$$

5. In a triangle $A = 35^\circ 20'$, $a = 127$, $b = 194$. Find B , C , and c .

INDEX

INDEX

Abscissa, 12.
Addition formulae, 155.
Altitude of sun, 89.
Ambiguous case, 120.
Angle, 25.
 circular measure of, 29.
 measurement of, 25.
 negative, 26.
 of elevation, 88.
 of depression, 88.
Antilogarithm, 21.
Arc, length of, 28.
 functions of, 38.
Area of triangle, 109.
Axes, 12.

Characteristic, 18.
Chord of angle, 40.
Circular functions, 40.
 measure, 29.
Complementary angle, ratios of, 40.
Co-ordinates, 12.
 cartesian, 12.
 polar, 41.
Cosecant, 38.
Cosine, 38.
 formula, 108.
 $\cos(A \pm B)$, 155.
 $\cos A \pm \cos B$, 159.
Cotangent, 38.
Course, 94.
Coversine, 40.

Degree, 25.
De Moivre's theorem, 152.
Departure, 94.
Difference of Latitude, 94.
 of two angles, ratios of, 156.
 of two sines, 159.
 of two cosines, 159.
Dip of the horizon, 127.
Double angle, ratios of, 100, 162.

Elementary formulae, 49.
Eliminant, 51.
Elimination, 51.

Equations, solution of, 78.
 solution by graphs, 17, 76.
 general solution of, 79.

Functions, algebraical, 14.
 circular, 40.
 trigonometrical, 40.

Geometry enunciations, 9.
Grade, 27.
Gradient, 33.
Graphs, general treatment, 12.
 of trigonometrical functions, 41, 163.
 harder trigonometrical, 74.
Greek alphabet, 24.

Half-angle, ratios of, 101, 104, 121, 162.
Haversine, 40, 121.
Heights and distances, 88, 124.

Identities, 50, 64, 167, 168, 170.
Imaginary quantities, 149.
Indices, fractional, 17.
 negative, 17.
Interpolation, 37.
Inverse functions, 45, 164.

Latitude, 32.
Logarithm, 18.
 base of, 18.
 use of, 22, 81.
 tabular, 36.
Longitude, 32.

Mantissa, 19.
Mariner's compass, 31.
Maximum value, 74.
Meridian, 32.
Minimum value, 74.

Navigation :
 plane sailing, 94.
 parallel sailing, 95.
 middle latitude sailing, 96.
 traverse sailing, 96.

- Negative angle, 26.
 ratios of, 54.
 direction, 13.
- Oral examples, 34, 61, 116, 123, 166.
- Ordinate, 12.
- Origin, 12.
- Parts of a triangle, 88.
- Point, 32.
- Points of the compass, 31.
- Powers of trigonometrical ratios, 44.
- Product of sines and cosines, 160.
- Projection, definition, 139.
 of area, 147.
 propositions, 140.
 length of, 40.
 formulae, 40, 109.
- Proportional parts, 71.
- Quadrilateral, area of cyclic, 111.
- Radian, 29.
- Radius of inscribed circle, 114, 168.
 of circumscribed circle, 114.
 of escribed circle, 114, 168.
- Rate of change of function, 76.
- Ratios, trigonometrical, 37.
 of $90 - A$, 53.
 of $180 - A$, 53.
 of $-A$, 54.
 of 0° and 90° , 56.
 of 30° , 45° , 60° , 57.
 of $2A$, 100, 162.
 of $\frac{1}{2}A$, 101, 162.
 of $3A$, 164.
 of $A + B$, 154.
 of $A - B$, 156.
 of $3A$, 164.
- Rhumb line, 94.
- Secant, 38.
- Sector, area of, 29.
- Sine, 38.
 formula, 107.
 graph, 41.
 $\sin(A \pm B)$, 154.
 $\sin A \pm \sin B$, 159.
- Slope of curve, 76.
- Small angle, 59.
 sine and tangent of, 60.
- Solution of right-angled triangle, 88.
 of triangles, 119.
 of equations, 78, 168.
 of equations by graphs, 17, 76.
- Sum of two angles, ratios of, 154.
 of two sines, 159.
 of two cosines, 159.
- Summation of a series, 169.
- Supplementary angle, ratios of, 53.
- Tabular logarithm, 36.
- Tangent, 38.
 graph, 41.
 $\tan(A \pm B)$, 163.
- Triangle formulae, 107.
 solution of, 119.
- Trigonometrical ratios:
 general definition of, 38.
 right-angled triangle, definition of, 86.
- Turning-points, 74.
- Vectors, 143.
 addition of, 144.
 resolution of, 145.
- Versine, 40.

$$b = 27.28 \text{ cm} \quad 25.39 \text{ cm}$$

$$1618^\circ 25'$$

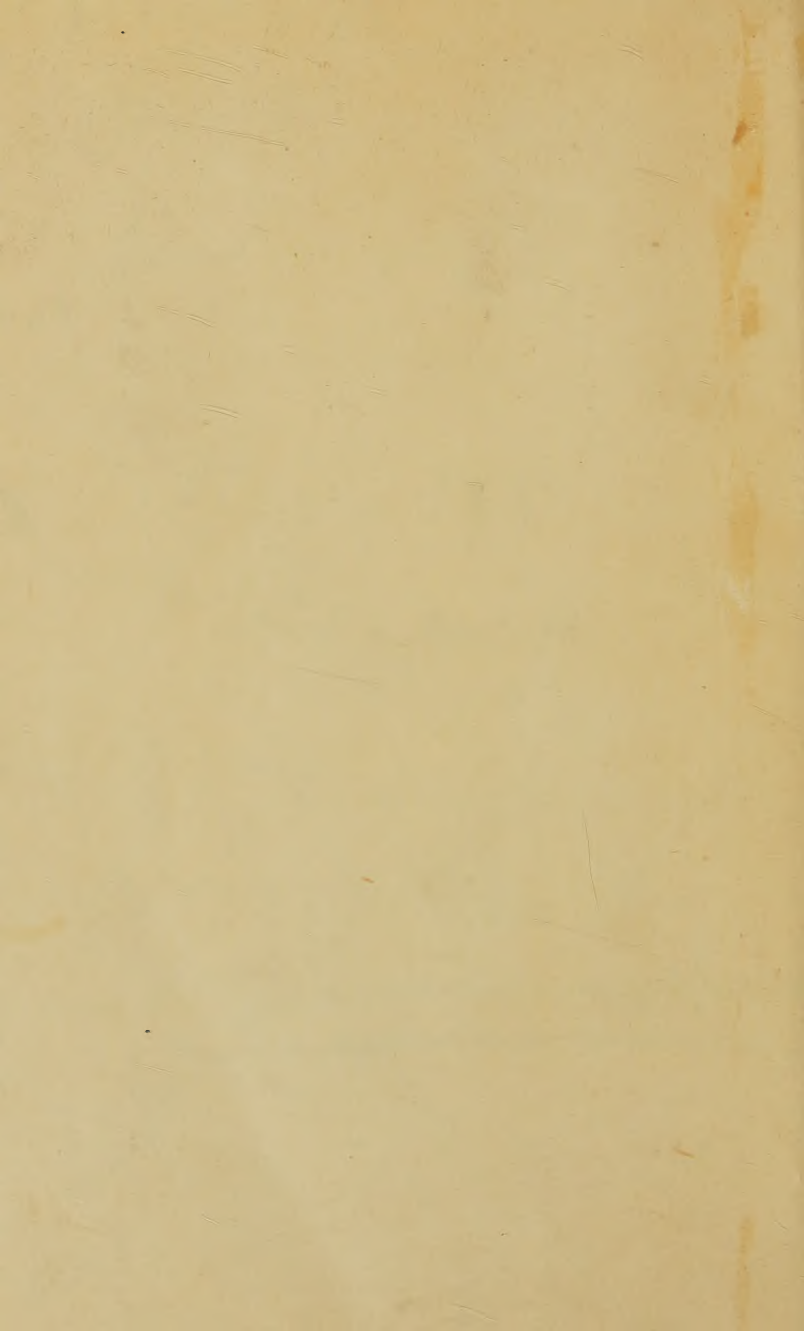
~~Sin~~

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$$



Arith X 111 219

3 4 6 7

184

18 (i) $\sin^{-1} \frac{4(2a)}{12}$
 (ii) $\tan^{-1} \frac{1}{2}$

19

20

21

(i) $\frac{\sin^{-1} L}{2a}$
 (ii) $\frac{\tan^{-1} L}{2a}$

22

(iv) $\frac{\sin^{-1} h \sin \theta}{2a}$
 or $\sin^{-1} R \sin \phi$

(v) $\tan^{-1} \left\{ a \sin \phi \div \left(\frac{1}{2} h a \cos \phi \right) \right\}$

10.18 cm

38° 30'

20 11° 16'

